RUIN PROBABILITIES IN TOUGH TIMES

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Part 1

HEAVY-TRAFFIC APPROXIMATION FOR FRACTIONALLY INTEGRATED RANDOM WALKS IN THE DOMAIN OF ATTRACTION OF A NONGAUSSIAN STABLE DISTRIBUTION

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Abstract. Motivated by applications to insurance mathematics, we prove some heavy-traffic limit theorems for process which encompass the fractionally integrated random walk as well as some FARIMA processes, when the innovations are in the domain of attraction of a nonGaussian stable distribution.

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1. Introduction and main result. The purpose of this paper is to study ruin probability when the claim process is nonstationary, has long range dependence, innovations in the domain of attraction of a stable distribution and when the premiums can barely cover the claims; hence the title.

The motivation for such a study, beyond the development of some of the mathematics needed to build more realistic models for some insurance companies, are manifold; this introduction seeks to describe them, relating the content of this paper to various problems considered before.

To start with, the claim processes we are interested in encompass many nonstationnary fractional autoregressive moving average (FARIMA) processes without prehistorical influence. As such, they also include the usual partial sum process. From an applied perspective, these processes are of interest because they are part of the standard models in time series, and their ontological justification as aggregation of simpler processes (Granger, 1980) has some appeal in economics and econometrics. From a theoretical perspective, their interest lies in the fact that they are not Markovian, may not have stationary solutions, may exhibit long range dependence, so may

not be amenable to the classical techniques, and yet are tractable. Consequently, a technical understanding of these models yields a greater understanding of the underlying stochastic phenomenon involved in simpler models. Indeed, as less technical tools become available, we have to resort to more fundamental aspects of the process involved. In that respect, the main contribution of this paper is threefold: firstly, it reveals the role of extreme values in fractional random walks with innovations in the domain of attraction of a non-Gaussian distribution; secondly, it gives a method of proof which, unlike all those we are aware of for the classical random walk, does not rely on either some form of Kolmogorov's maximal inequality or the Wiener-Hopf factorization — see a discussion of the classical proofs in Shneer and Wachtal (2009); thirdly, it shows that the so-called exponential representation of uniform order statistics may be used in sequential problems, even though this representation is nonsequential in nature.

A certain number of results known for the partial sum process have been extended to its fractionally integrated version, and, more generally, to FARIMA ones. For instance, motivated by applications in econometrics, Donsker's (1951) invariance principle, asserting the convergence of the rescaled partial sum process to a Wiener process, has been extended to some FARIMA processes by Philipps (1987) and Akonom and Gouriéroux (1987); the latter authors showed that a fractional integral of the Wiener process, that is, a fractional Brownian motion, may arise as limiting process — see Wu and Shao (2006) for extensions and further references. In a similar spirit, Barbe and Broniatowski (1998) extented Varadhan's (1966) large deviation result for partial sums to some FARIMA processes — see also Ghosh and Samorodnitsky (2009) for related results in the stationary case. In that extension, the derivative involved in Varadhan's action functional was replaced by a fractional derivative. The classical ruin estimate of Cramér has been partially extended by Barbe and McCormick (2008b); and its heavy tail analogue, Veraverbeke's (1977) Theorem 2, has been also studied in the setting of FARIMA processes by Barbe and McCormick (2008a). The general thrust of these works is to understand how classical results for partial sums extend to their fractional analogue, somewhat paralleling the developments related to fractional Brownian motion and fractional Lévy processes in probability.

In the classical applied probability area of queueing theory, a well

studied topic is that of the so-called heavy traffic approximations, referring to an asymptotic analysis of the behavior of queues with traffic intensity near one. In an insurance mathematics setting, this translates into studying a risk process when the premium can barely cover the claims. For the partial sum process, heavy traffic approximations are well understood and a presentation in book form may be found in Resnick (2007) — see also Shneer and Wachtel (2009) as well as Kosiński, Boxma and Zwart (2010) for further references and results, and Whitt (2002) for examples of applications in queueing theory. One of the purposes of this paper is to present an analogous result in the setting of FARIMA processes.

In yet another direction which we will not pursue, heavy traffic approximation can be interpreted in terms of moving boundary crossing probability. In particular, our result can give the probability that a fractional random walk, or more generaly, a FARIMA process without prehistorical influence, crosses a moving curved boundary.

Throughout this paper we use the letter c for a generic constant whose value may change from one occurence to another.

We use the symbol $\lesssim$ between two sequences, as in say $a_n \lesssim b_n$, to signify that $a_n \leqslant b_n\bigl(1+o(1)\bigr)$ as n tends to infinity.

2. Main result. The processes which we will be dealing with are defined through an analytic function g on $(-1,1)$ and a distribution function F on the real line. These two pieces of data allow us to build a so-called (g,F)-process as follows. Consider a sequence $(X_i)_{i\geqslant 1}$ of independent random variables, all having F for distribution function. We consider the series expansion of g,

$$g(x) = \sum_{i\geqslant 0} g_i x^i \,.$$

A (g,F)-process $(S_n)_{n\geqslant 0}$ is defined by $S_0=0$ and

$$S_n = \sum_{0\leqslant i<n} g_i X_{n-i}\,, \qquad n\geqslant 1\,.$$

Setting $X_i=0$ if i is negative, and writing B for the backward shift operator acting on sequences, that is $BX_i = X_{i-1}$, we see that the above expression for S_n amounts to

$$S_n = g(B)X_n\,.$$

If $g(x) = 1/(1-x)$, then (S_n) is the partial sum process of the X_i. If g is a rational function continuous on $[\,-1,1\,]$, then a (g,F)-process is an ARMA one. If g is $(1-\mathrm{Id})^{-d}$ times a rational function continuous on $[\,-1,1\,]$, then a (g,F)-process is a FARIMA one.

Introducing the notation

$$g_{[0,n)} = \sum_{0\leqslant i<n} g_i\,,$$

we see that if F has finite mean, then $S_n = g(B)(X_n - \mathrm{E}X_n) + g_{[0,n)}\mathrm{E}X_1$. If the sequence (g_n) is ultimately positive and not summable, then S_n drifts to $+\infty$ if $\mathrm{E}X_1$ is positive and $-\infty$ if $\mathrm{E}X_1$ is negative. Our heavy traffic approximation yields the limiting behavior of $\max_{n\geqslant 0} S_n$ as the expected value of X_1 tends to 0 from below. This amounts to assuming that the innovations are centered and seek the asymptotic behavior of $\max_{n\geqslant 0}(S_n - ag_{[0,n)})$ as a tends to 0 from above. As mentioned in the introduction, this can be interpreted as a problem on moving boundary crossing, for the inequality $\max_{n\geqslant 0}(S_n - ag_{[0,n)}) > x$ is equivalent to the process S_n crossing the boundary $x + ag_{[0,n)}$.

A simple examination of known heavy traffic approximation results for the classical random walk shows that different asymptotic behaviors are to be expected according to the tail behavior of the innovation, and, in particular, according to the finiteness of the variance. The invariance principle of Barbe and McCormick (2010) also suggests that one should distinguish the cases where the sequence (g_n) diverges to infinity and that where it tends to 0, corresponding respectively to a fractional integration and differentiation of the random walk. In this paper, we will concentrate on the fractional integration; a companion paper deals with the fractional differentiation. This discussion, as well as technical requirements for the proof lead us to introduce some assumptions.

The summability of the g_i is related to the behavior of g near -1 and 1, and bears on the long range dependence properties of the process. We will restrict ourselves to what Granger (1988) called the generalized integrated processes, assuming that

$$g(1-1/\mathrm{Id}) \text{ is regularly varying of positive index } \gamma. \tag{2.1}$$

Karamata's theorem for power series asserts that if (g_n) is asymptotically equivalent to a monotone sequence, then (2.1) is equivalent to (g_n) being regularly varying of index $\gamma - 1$. In this case,

$g_i/g_n \sim (i/n)^{\gamma-1}$ as n tends to infinity and i/n stays bounded away for the origin and infinity. We will assume more; firstly, that

$$(g_n) \text{ is normalized regularly varying,} \tag{2.2}$$

meaning that (see Bingham, Goldie and Teugels, 1989, Theorem 1.9.8)

$$\frac{g_{n+1}}{g_n} = 1 + \frac{\gamma-1}{n}\big(1+o(1)\big)$$

as n tends to infinity, and, secondly, that there exists a positive δ such that

$$\lim_{n\to\infty} n^\delta \sup_{n^{-\delta}\leqslant i/n\leqslant 1} \Big| \frac{g_i}{g_n} - \Big(\frac{i}{n}\Big)^{\gamma-1} \Big| = 0\,. \tag{2.3}$$

We will prove that assumptions (2.2) and (2.3) hold for FARIMA processes; therefore, they are not overwhelmingly restrictive in applications. Note that (2.3) implies (2.1) and that there is no loss of generality to assume that $\delta < 1/2$; indeed, if (2.3) holds for some δ, then it holds for any smaller one. Furthermore, (2.2) implies (2.1) as well.

As far as the innovations are concerned, we assume that they have a mean but no variance, and, more precisely, that

$$\begin{gathered} F \text{ is centered and in the domain of attraction of a} \\ \text{stable distribution of index } \alpha \text{ in } (1,2). \end{gathered} \tag{2.4}$$

Whenever G is a cummulative distribution function, we write $\overline{G}$ for $1-G$. We write F_* for the distribution function of $|X_1|$. Assumption (2.4) implies that one of the tails of F is regularly varying of index $-\alpha$ and that F is tail balanced, meaning the following. Write $\mathsf{M}_{-1}F$ for the distribution function of $-X$. Then $\overline{F}_*$ coincides with $\overline{F} + \overline{\mathsf{M}_{-1}F}$ on the positive half-line when F is continuous. The tail balance condition is that $\overline{F} \sim p\overline{F}_*$ and $\overline{\mathsf{M}_{-1}F} \sim q\overline{F}_*$ at infinity where p and q are nonnegative numbers which add to 1. For simplicity, we consider throughout the paper and without mentioning it any further that p does not vanish.

Writing

$$F^{\leftarrow}(u) = \inf\{\, x : F(x) \geqslant u \,\}$$

for the càglàd quantile function associated to F, (2.4) implies that $F^{\leftarrow}(1-1/\mathrm{Id})$ is regularly varying of index $1/\alpha$; if q does not vanish,

$(\mathsf{M}_{-1}F)^{\leftarrow}(1-1/\mathrm{Id})$ is regularly varying of the same index $1/\alpha$. Paralleling (2.3), we assume that for some positive κ,

$$\lim_{t\to\infty} t^{\kappa} \sup_{t^{-\kappa}\leqslant\lambda\leqslant t^{\kappa}} \left| \frac{F^{\leftarrow}(1-\lambda/t)}{F^{\leftarrow}(1-1/t)} - \lambda^{-1/\alpha} \right| = 0\,. \qquad (2.5)$$

If this assumption holds for some κ, then it holds for any smaller one. While this assumption is stated in a form convenient for our usage, its meaning is made more explicit in the following result, whose proof is deferred to section 4.

Proposition 2.1. *Let κ be a positive real number less than both 1 and $2/(\alpha+1)$. The following are equivalent as t tends to infinity:*

(i) $t^{\kappa} \sup_{t^{-\kappa}\leqslant\lambda\leqslant t^{\kappa}} \left| \frac{F^{\leftarrow}(1-\lambda/t)}{F^{\leftarrow}(1-1/t)} - \lambda^{-1/\alpha} \right| = o(1)$.

(ii) $F^{\leftarrow}(1-1/t) = ct^{1/\alpha}\big(1+o(t^{-\kappa(1+1/\alpha)})\big)$.

Moreover, if F is continuous and increasing, it is also equivalent to

(iii) $\overline{F}(t) = (c/t)^{\alpha}\big(1+o(t^{-\kappa(\alpha+1)})\big)$.

It is likely that our need for (2.5) is an artifact of the technique used in the proof, and that our result holds in a much greater generality. This issue is discussed after the proof, in section 3.6.

Considering $(1/g_n)$, Proposition 2.1 implies that condition (2.3) is equivalent to the existence of a positive ϵ such that

$$g_n = cn^{\gamma-1}\big(1+o(n^{-\epsilon})\big)$$

as n tends to infinity.

Concerning the lower tail we will assume either an analogue of (2.5), namely

$$\lim_{t\to\infty} t^{\kappa} \sup_{t^{-\kappa}\leqslant\lambda\leqslant t^{\kappa}} \left| \frac{(\mathsf{M}_{-1}F)^{\leftarrow}(1-\lambda/t)}{(\mathsf{M}_{-1}F)^{\leftarrow}(1-1/t)} - \lambda^{-1/\alpha} \right| = 0\,, \qquad (2.6)$$

an assumption which is relevant when q does not vanish and in some cases when q vanishes, or, forcing q to vanish,

$$\overline{\mathsf{M}_{-1}F}(t) \leqslant c\overline{F}(t\log t)\log t \quad \text{ultimately.} \qquad (2.7)$$

Note that while assumptions (2.6) and (2.7) do not cover all possible distributions, it is not much more restrictive than (2.5) in practical

applications; indeed all classical distributions which satisfy (2.5) satisfy either (2.6) or (2.7).

The distribution function F yields the Lévy measure ν, defined by its density with respect to the Lebesgue measure λ,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x) = p\alpha x^{-\alpha-1}\mathbb{1}_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1}\mathbb{1}_{(-\infty,0)}(x)\,.$$

It induces a Lévy stable process L_0 with Lévy measure ν, that is a process with selfsimilar and independent increments, such that, under (2.4),

$$\mathrm{E}e^{itL_0(1)} = \exp\Bigl(\int (e^{itx} - 1 - itx)\,\mathrm{d}\nu(x)\Bigr)\,.$$

The subscript 0 is to indicate that this process is centered. A fractional Lévy stable process is defined through the Riemann-Liouville integral

$$L_0^{(\gamma-1)}(t) = \gamma\int_0^t (t-u)^{\gamma-1}\,\mathrm{d}L_0(u)\,.$$

We will use the function

$$k = \frac{\mathrm{Id}}{F_*^{\leftarrow}(1-1/\mathrm{Id})}\,.$$

It is regularly varying of positive index $1-1/\alpha$.

Our main result is the following.

Theorem 2.2. *Assume that γ is greater than 1 and that (2.2), (2.3), (2.4) and (2.5) hold. If either (2.6) or (2.7) hold, then*

$$\lim_{a\to 0}\frac{1}{ag\bigl(1-1/k^{\leftarrow}(1/a)\bigr)}\sup_{n\geqslant 1}(S_n - ag_{[0,n)}) \stackrel{\mathrm{d}}{=} \sup_{t\geqslant 0}\bigl(L_0^{(\gamma-1)}(t) - t^\gamma\bigr)\,.$$

Moreover, the random variable $\sup_{t\geqslant 0}\bigl(L_0^{(\gamma-1)}(t) - t^\gamma\bigr)$ is almost surely finite.

Example. We consider a FARIMA process without prehistorical influence defined as follows. Let Θ and Φ be two real polynomials, the roots of Φ being outside the complex unit disk and Θ not vanishing at 1. The FARIMA process

$$(1-B)^\gamma\Phi(B)S_n = \Theta(B)X_n$$

is a (g, F)-process with $g = (1-\mathrm{Id})^{-\gamma}\Theta/\Phi$. We assume that γ is greater than 1. This process is then a fractionally integrated random walk.

Lemma 6.1 in Barbe and McCormick (2010) implies that assumption (2.2) holds. Checking that (2.3) holds is easy, because Lemmas 6.1 and 6.2 in Barbe and McCormick (2010) show that when g is $(1-\mathrm{Id})^{-\gamma}\Theta/\Phi$, there exists a converging sequence (a_n) such that

$$g_n = cn^{\gamma-1}\Big(1+\frac{a_n}{n}\Big) .$$

This implies that whenever i and n tend to infinity with i at most n,

$$\begin{aligned} n^\delta\Big|\frac{g_i}{g_n}-\Big(\frac{i}{n}\Big)^{\gamma-1}\Big| &= n^\delta\Big(\frac{i}{n}\Big)^{\gamma-1}\Big|\frac{1+a_i/i}{1+a_n/n}-1\Big| \\ &\lesssim n^\delta\Big|\frac{a_i}{i}-\frac{a_n}{n}\Big| . \end{aligned} \tag{2.8}$$

If i is in the range $[\,n^{1-\delta}, n\,]$, then n^δ/i tends to 0 whenever δ is less than 1/2, and, similarly, n^δ/n tends to 0 as n tends to infinity. Since (a_n) converges, we see that (2.8) implies (2.3).

To fix the ideas, consider a distribution function F such that $\overline{F}(x) \sim cx^{-\alpha}$. Then $F^{\leftarrow}(1-1/t) \sim (ct)^{1/\alpha}$ as t tends to infinity. Since $F_*^{\leftarrow}(1-1/t) \sim p^{-1/\alpha}F^{\leftarrow}(1-1/t)$, we obtain

$$k(t) \sim (p/c)^{1/\alpha} t^{(\alpha-1)/\alpha}$$

as t tends to infinity. It follows that

$$k^{\leftarrow}(1/a) \sim (c/p)^{1/(\alpha-1)} a^{-\alpha/(\alpha-1)}$$

as a tends to 0. Since $g(1-1/x) \sim x^\gamma\Theta(1)/\Phi(1)$ as x tends to infinity, we have

$$ag\Big(1-\frac{1}{k^{\leftarrow}(1/a)}\Big) \sim (c/p)^{\gamma/(\alpha-1)} a^{(\alpha-1-\alpha\gamma)/(\alpha-1)}\frac{\Theta(1)}{\Phi(1)}$$

as a tends to 0. Hence, assuming that (2.5) holds, we obtain that

$$\sup_{n\geqslant 0}(S_n - ag_{[0,n)}) \sim (c/p)^{\gamma/(\alpha-1)}\frac{\Theta(1)}{\Phi(1)} a^{1-\gamma\alpha/(\alpha-1)} \sup_{t\geqslant 0}\big(L_0^{(\gamma-1)}(t) - t^\gamma\big)$$

as a tends to 0. In particular, the left hand side grows like $a^{1-\gamma\alpha/(\alpha-1)}$. It is interesting to note that the exponent involved depends on α and γ only through $\gamma\alpha/(\alpha-1)$, that is γ times the conjugate exponent of α.

3. Proof of Theorem 2.2. For the classical random walk, there exists two ways of proving a heavy traffix approximation: one based on the Wiener-Hopf factorization proposed by Kingman (1961, 1962, 1965), one based on a functional limit theorem proposed by Prohorov (1963). We follow Prohorov's approach.

Throughout the proof we will use many times the following form of Karamata's theorem for power series (see Bingham, Goldie and Teugels, 1989, Corollary 1.7.3). If (g_n) is regularly varying of positive index $\gamma-1$, it is asymptotically equivalent to an increasing sequence and

$$g_n \sim \frac{\gamma g_{[0,n)}}{n} \sim \frac{g(1-1/n)}{n\Gamma(\gamma)} \tag{3.1}$$

as n tends to infinity.

To proceed with the proof, up to an asymptotic equivalence, define $\Lambda = \Lambda(1/a)$ by the relation

$$ak(\Lambda) \sim 1 \tag{3.2}$$

as a tends to 0. It follows from Barbe and McCormick's (2010) Theorem 5.2 that, in the sense of weak$*$ convergence of distribution of stochastic processes in $\mathrm{D}[\,0,\infty)$ endowed with the topology of uniform convergence on compactas,

$$\frac{k(\Lambda)}{g_{[0,\Lambda)}} S_{\lfloor \Lambda \mathrm{Id} \rfloor} \xrightarrow{\mathrm{d}} L_0^{(\gamma-1)}$$

as Λ tends to infinity. Since (3.1) and (2.3) imply that $(g_{[0,n)})$ is a regularly varying sequence of index γ, (3.2) implies that we have the convergence of stochastic processes

$$\frac{k(\Lambda)}{g_{[0,\Lambda)}} \big(S_{\lfloor \Lambda \mathrm{Id} \rfloor} - a g_{[0,\Lambda \mathrm{Id})}\big) \xrightarrow{\mathrm{d}} L_0^{(\gamma-1)} - \mathrm{Id}^\gamma\,. \tag{3.3}$$

Consequently, for any positive T,

$$\frac{k(\Lambda)}{g_{[0,\Lambda)}} \sup_{0\leqslant n\leqslant \Lambda T} (S_n - a g_{[0,n)}) \xrightarrow{\mathrm{d}} \sup_{0\leqslant t\leqslant T} \big(L_0^{(\gamma-1)}(t) - t^\gamma\big)$$

as a tends to 0.

Since $\sup_{0\leqslant t\leqslant T}\bigl(L_0^{(\gamma-1)}(t)-t^\gamma\bigr)$ is nondecreasing in T, it converges almost surely as T tends to infinity, possibly to infinity. Hence, to prove Theorem 2.2, it suffices to show that

$$\lim_{T\to\infty}\limsup_{a\to 0}\mathrm{P}\{\,\exists n>\Lambda T\,:\,S_n>ag_{[0,n)}\,\}=0 \tag{3.4}$$

and $\sup_{t\geqslant 0}\bigl(L_0^{(\gamma-1)}(t)-t^\gamma\bigr)$ is almost surely finite.

As pointed out by Shneer and Wachtel (2009), or differently in Szczotka and Woykzyński (2003), the main difficulty in proving a heavy traffic approximation for sums is to show that the maximum of the process does not occur too far in time, that is, in our case proving (3.4). In the context of (g,F)-processes this task is far more involved than for ordinary random walks, mostly because there is no analogue of Kolmogorov's maximal inequality. In order to explain our proof, that is, the remainder of this paper, we need to make a preliminary study of (3.4).

Given (3.2), we see that (3.4) is equivalent to

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty}\mathrm{P}\{\,\exists n>\Lambda T\,:\,S_n>g_{[0,n)}/k(\Lambda)\,\}=0\,. \tag{3.5}$$

Substituting Λ for ΛT, using that

$$k(\Lambda/T)\sim T^{(1/\alpha)-1}k(\Lambda)$$

as Λ tends to infinity, and using that $1-1/\alpha$ is positive, substituting T for $T^{1-1/\alpha}$, we obtain that (3.5) is equivalent to

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty}\mathrm{P}\{\,\exists n>\Lambda\,:\,S_n>Tg_{[0,n)}/k(\Lambda)\,\}=0\,. \tag{3.6}$$

We can now explain how to prove Theorem 2.2. The proof has four main steps. The first two aim at showing that instead of considering all n exceeding Λ in (3.6), we can reduce the range to all n between Λ and $\Lambda^{1+\epsilon}$, where ϵ is positive but can be chosen as one wishes. This is achieved by showing in the first step that the innovations coming from the central part of the distribution can be ignored. In the second step, a simple bound on the contribution of the largest innovations permits us to show that if the event involving S_n in (3.6) occurs, it is very likely that n is less than $\Lambda^{1+\epsilon}$. Being able to concentrate on the range of n between Λ and $\Lambda^{1+\epsilon}$, the third step

consists in showing a result similar to that of the first step, namely that the innovations not too large can be ignored; while this mid-range depends on n in the first step, this dependence will be, in some sense, less so in the third step. The fourth step, by far the most complicated, consists in showing that the contribution of the extreme innovations to S_n, properly rescaled, can be approximated by a fractional Lévy process, uniformly in the range of n between Λ and $\Lambda^{1+\epsilon}$. While this shares some similarity with (3.3) and will be proved with a technique inspired by our proof of (3.3), this is more difficult than proving (3.3). The reason is that in (3.3) we approximate the process S_n on $[\,0,\Lambda\,]$, which is Λ times the fixed compact set $[\,0,1\,]$. In contrast in our problem, the set $[\,\Lambda,\Lambda^{1+\epsilon}\,]$ should be thought as Λ times $[\,1,\Lambda^{\epsilon}\,]$, that is Λ times an interval whose length diverges with Λ. This forces us to develop a sequential analogue of the representation used in Barbe and McCormick (2010), and it is likely that the technique used will be of value for related boundary crossing problems. A fifth step concludes the proof, mostly taking care of the lower tail of the distribution and doing some bookeeping.

3.1. Step 1. We first consider the part of the process (S_n) made from the not too large innovations. In order to set up the proper thresholdings, let (a_n) and (b_n) be two sequences of real numbers such that

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{and} \qquad \lim_{n\to\infty} b_n = +\infty\,.$$

Since F obeys a tail balance condition and we suppose that p does not vanish, it is convenient to assume that

$$\lim_{n\to\infty} b_n/(-a_n) \quad \text{is positive or infinite.}$$

We define the variance of the truncated innovations,

$$\sigma_n = \mathrm{Var}\bigl(X\mathbb{1}_{[a_n,b_n]}(X)\bigr)\,,$$

and the centered and standardized 'middle' innovations,

$$Z_{i,n} = \frac{X_i\mathbb{1}_{[a_n,b_n]}(X_i) - \mathrm{E}X\mathbb{1}_{[a_n,b_n]}(X)}{\sigma_n}\,, \qquad 1\leqslant i\leqslant n\,.$$

The 'middle' part of S_n is then

$$M_n = \sigma_n \sum_{0\leqslant i<n} g_i Z_{n-i,n} \,.$$

Given that we use (a_n) and (b_n) to truncate the innovations and that we will use a quantile transformation, it is convenient, as in Barbe and McCormick (2010), to take both sequences as quantiles. Since both sequences are constructed in similar fashion, we explain that of (b_n). We consider a sequence $(\widetilde{m}_n)$ which is regularly varying of positive index β less than 1. We set $\widetilde{b}_n = F^{\leftarrow}(1-\widetilde{m}_n/n)$. Setting $m_n = n\overline{F}(\widetilde{b}_n)$, the sequence

$$(m_n) \text{ is regularly varying of index } \beta$$

as well. We then take

$$b_n = F^{\leftarrow}(1-m_n/n)\,.$$

Since (m_n) is a regularly varying sequence of index β, the sequence (b_n) is regularly varying of index $(1-\beta)/\alpha$.

This construction ensures that (m_n) is regularly varying of index β and $1-m_n/n$ is in the range of F. This ensures that the inequality $F^{\leftarrow}(1-u) > b_n$ is equivalent to $u < m_n/n$ (see Shorack and Wellner, 1986, §1, pp. 5–7). It is implicit that a similar construction is made for (a_n), switching the tails.

In order to avoid heavy subscripts and many integer parts brackets, we will sometimes use the function $m(\cdot)$ defined by $m(x) = m_{\lfloor x\rfloor}$. We will also write m_x for $m_{\lfloor x\rfloor}$.

Our next proposition asserts that the middle part can be neglected in our problem. Recall that the parameter β regulates the growth of our truncation sequence used to define M_n.

Proposition 3.1.1. *For any β positive and less than 1, for any positive T,*

$$\lim_{\Lambda\to\infty} \mathrm{P}\{\, \exists n \geqslant \Lambda \,:\, M_n > T g_{[0,n)}/k(\Lambda)\,\} = 0\,.$$

Proof. Lemma 2.1.1 in Barbe and McCormick (2010) asserts that

$$\sigma_n \sim c b_n \sqrt{\overline{F}(b_n)} \sim c F^{\leftarrow}(1-m_n/n)\sqrt{m_n/n}$$

as n tends to infinity. Inequality (2.2.1) in Barbe and McCormick (2010) implies that for any positive integer r there exists a constant c_r such that for any positive n,

$$\begin{aligned}\Big|\mathrm{E}\Big(\frac{M_n}{\sigma_n\sqrt{n}}\Big)^r\Big| &\leqslant \frac{c_r}{n}\sum_{1\leqslant i\leqslant n}|g_i|^r\\ &\sim c_r|g_n|^r\int_0^1 u^{r(\gamma-1)}\,\mathrm{d}u\,,\end{aligned}$$

the asymptotic equivalence being as n tends to infinity. Using Markov's inequality and (3.1), for any positive integer r,

$$\begin{aligned}\mathrm{P}\{\,M_n > Tg_{[0,n)}/k(\Lambda)\,\} &\leqslant cg_n^r\Big(\frac{k(\Lambda)\sigma_n\sqrt{n}}{Tg_{[0,n)}}\Big)^r\\ &\leqslant c\Big(\frac{k(\Lambda)}{Tn}F^{\leftarrow}(1-m_n/n)\sqrt{m_n}\Big)^r\\ &\sim \frac{c}{T^r}m_n^{-r/2}\Big(\frac{k(\Lambda)}{k(n/m_n)}\Big)^r \qquad (3.1.1)\end{aligned}$$

as Λ tends to infinity and uniformly in $n\geqslant\Lambda$. Since α is less than 2, let η be a positive real number so that $(1/2)-(1/\alpha)+\eta$ is negative. Potter's bound implies

$$k\Big(\frac{n}{m_n}\Big)\gtrsim m_n^{(1/\alpha)-1-\eta}k(n)\,.$$

It follows that (3.1.1) is of order at most

$$\frac{c}{T^r}m_n^{r((1/2)-(1/\alpha)+\eta)}\Big(\frac{k(\Lambda)}{k(n)}\Big)^r\,.$$

For T and Λ large enough, this bound is less than 1 since $(1/2)-(1/\alpha)+\eta$ is negative, r is positive, and k is regularly varying of positive index. Applying Bonferroni's inequality, we obtain

$$\begin{aligned}&\mathrm{P}\{\,\exists n\geqslant\Lambda\,:\,M_n > Tg_{[0,n)}/k(\Lambda)\,\}\\ &\qquad\leqslant \frac{c}{T}m_\Lambda^{r((1/2)-(1/\alpha)+\eta)}k(\Lambda)^r\sum_{n\geqslant\Lambda}k(n)^{-r}\,. \qquad (3.1.2)\end{aligned}$$

Taking r greater than $\alpha/(\alpha-1)$ ensures that the series of generic term $k(n)^{-r}$ converges and that $\sum_{n\geqslant\Lambda}k(n)^{-r}$ is of order $\Lambda k(\Lambda)^{-r}$ as

Λ tends to infinity. In this case, (3.1.2) is of order $m_\Lambda^{r((1/2)-(1/\alpha)+\eta)}\Lambda$. This bound is regularly varying in Λ of index $\beta r\bigl((1/2)-(1/\alpha)+\eta\bigr)+1$. This index is negative whenever r is large enough. ∎

3.2. Step 2. We consider the contribution of the extreme innovations to S_n,

$$T_n^+ = \sum_{0\leqslant i<n} g_i X_{n-i}\mathbb{1}_{(b_n,\infty)}(X_{n-i})\,.$$

In order to understand precisely the role of (2.5) we introduce a slight refinment. Let ξ be an ultimately increasing slowly varying function, diverging to infinity at infinity, such that $\xi(n^2)\sim c\xi(n)$ as n tends to infinity, and

$$\sum_{n\geqslant 1}\frac{1}{n\xi(n)}<\infty\,. \tag{3.2.1}$$

One could take $\xi(x)$ to be $(\log x)^{1+\eta}$ or $(\log x)(\log\log x)^{1+\eta}$ for some positive η; one may simply replace $\xi(n)$ by $\log^2 n$ when reading the remainder of the proof. However, having introduced the function ξ will allow us to understand the role of (2.5). Sometimes we will write ξ_n instead of $\xi(n)$. The key requirement, (3.2.1), is equivalent to the assertion that the smallest of n independent random variables uniformly distributed over $[\,0,1\,]$ is greater than $1/(n\xi_n)$ almost surely for n large enough (see Geffroy, 1958, 1959, and Kiefer, 1972); the other conditions are of technical nature.

We introduce the following variant of (2.5): there exists a real number ρ greater than 1 such that

$$\limsup_{n\to\infty} m_n^\rho \sup_{1/m_n\leqslant\lambda\leqslant m_n}\Bigl|\frac{F^{\leftarrow}(1-\lambda/n)}{F^{\leftarrow}(1-1/n)}-\lambda^{-1/\alpha}\Bigr|<\infty\,, \tag{3.2.2}$$

as well as the condition

$$\liminf_{n\to\infty} m_n/\xi_n>0\,. \tag{3.2.3}$$

Note that if (m_n) is regularly varying of positive index β less than κ, then (2.5) implies (3.2.2) and (3.2.3); indeed, (2.5) the implies

$$\lim_{n\to\infty} n^\kappa \sup_{1/m_n\leqslant\lambda\leqslant m_n}\Bigl|\frac{F^{\leftarrow}(1-\lambda/n)}{F^{\leftarrow}(1-1/n)}-\lambda^{-1/\alpha}\Bigr|=0\,,$$

and, considering the indices of regular variation, we can take ρ to be any number greater than 1 and less than κ/β. In what follows will rely solely on the combination (3.2.2) and (3.2.3), and, except specified otherwise, not on the positivity of the index of regular variation β of (m_n). In particular, if β is allowed to vanish, (m_n) is allowed to be slowly varying. Ultimately, this will inform us on the role of (2.5). A further discussion is presented in section 3.6.

Proposition 3.2.1. *Let ϵ be a positive real number. If*

$$\beta < \Bigl(\frac{\epsilon}{1+\epsilon}\frac{\alpha-1}{\alpha}\Bigr) \wedge \kappa .$$

then for any positive T,

$$\lim_{\Lambda\to\infty} \mathrm{P}\{\, \exists n \geqslant \Lambda^{1+\epsilon} \,:\, |T_n^+ - \mathrm{E}T_n^+| \geqslant T g_{[0,n)}/k(\Lambda)\,\} = 0 .$$

In order to prove Proposition 3.2.1, we need the following estimates on the expectation $\mu_n^+ = \mathrm{E}X\mathbb{1}_{[b_n,\infty)}(X)$.

Lemma 3.2.2. *If (3.2.2) holds, then*

$$\limsup_{n\to\infty} \Bigl| k(n)\mu_n^+ - \frac{\alpha}{\alpha-1} m_n^{1-1/\alpha} \Bigr| < \infty .$$

Proof. Let U be a uniform random variable over $[\,0,1\,]$. Thinking of the random variable X as $F^{\leftarrow}(1-U)$,

$$\begin{aligned}\frac{n\mu_n^+}{F^{\leftarrow}(1-1/n)} &= n\int_0^{m_n/n} \frac{F^{\leftarrow}(1-u)}{F^{\leftarrow}(1-1/n)}\,\mathrm{d}u\\ &= \int_0^{m_n} \frac{F^{\leftarrow}(1-v/n)}{F^{\leftarrow}(1-1/n)}\,\mathrm{d}v .\end{aligned}$$

Assumption (3.2.2) yields

$$\begin{aligned}\int_{1/m_n}^{m_n} \frac{F^{\leftarrow}(1-v/n)}{F^{\leftarrow}(1-1/n)}\,\mathrm{d}v &= \int_{1/m_n}^{m_n} v^{-1/\alpha} + O(m_n^{-1})\,\mathrm{d}v\\ &= \frac{\alpha}{\alpha-1} m_n^{1-1/\alpha} + O(1) .\end{aligned}$$

Furthermore, Potter's bound implies that for any η positive less than $(\alpha-1)/\alpha$, as n tends to infinity,

$$\int_0^{1/m_n} \frac{F^{\leftarrow}(1-v/n)}{F^{\leftarrow}(1-1/n)}\,\mathrm{d}v \leqslant 2\int_0^{1/m_n} v^{-(1/\alpha)-\eta}\,\mathrm{d}v = O(1)$$

This proves the lemma. ∎

Note that Lemma 3.2.2 implies, as n tends to infinity,

$$\mu_n^+ \sim \frac{\alpha}{\alpha-1}\frac{m_n^{1-1/\alpha}}{k(n)} \tag{3.2.4}$$

Proof of Proposition 3.2.1. The proof has two steps. In the first one we prove that, almost surely, T_n^+ cannot exceed $g_{[0,n)}/k(\Lambda)$ whenever n exceeds $\Lambda^{1+\epsilon}$ and Λ is large enough. In the second one, we prove a similar assertion on the expectation $\mathrm{E}T_n^+$. Recall that since (g_n) is regularly varying of positive index, it is asymptotically equivalent to a nondecreasing sequence.

Step 1. Let $(U_i)_{i\geqslant 1}$ be a sequence of independent random variables, uniformly distributed on $[\,0,1\,]$. There is no loss of generality in assuming that $X_i = F^{\leftarrow}(1-U_i)$. Since we use the càglàd version of the quantile function and $1-m_n/n$ is in the range of F, the inequality $F^{\leftarrow}(1-U) > F^{\leftarrow}(1-m_n/n)$ occurs if and only if $U < m_n/n$ (see Shorack and Wellner, 1986, §I.1, pp.5–7). Therefore, writing $\mathbb{U}_n$ for the empirical distribution function of $(U_i)_{1\leqslant i\leqslant n}$, we have, for any n large enough,

$$\begin{aligned} T_n^+ &= \sum_{0\leqslant i<n} g_i F^{\leftarrow}(1-U_{n-i})\mathbb{1}\{\,U_i \leqslant m_n/n\,\} \\ &\leqslant 2g_n F^{\leftarrow}(1-U_{1,n})n\mathbb{U}_n(m_n/n)\,. \end{aligned} \tag{3.2.5}$$

From Theorem 1 in Kiefer (1972) we deduce that $U_{1,n} \geqslant 1/n\xi_n$ almost surely for n large enough, and from Theorem 2 in Shorack and Wellner (1978), we conclude that $\mathbb{U}_n \leqslant \xi_n\mathrm{Id}$ almost surely for n large enough. Therefore, using (3.1), (3.2.5) is ultimately at most

$$2g_nF^{\leftarrow}\Bigl(1-\frac{1}{n\xi_n}\Bigr)m_n\xi_n \sim cg_{[0,n)}\frac{m_n\xi_n^2}{k\bigl(n\xi_n\bigr)}\,. \tag{3.2.6}$$

Recall that $m(\cdot)$ is the function such that $m(x) = m_{\lfloor x \rfloor}$. Provided β is less than $1 - 1/\alpha$, the function $m_n \xi_n^2 / k(n\xi_n)$ is regularly varying in n of negative index $\beta - 1 + (1/\alpha)$. Thus, (3.2.6) is at most

$$c g_{[0,n)} \frac{m(\Lambda^{1+\epsilon}) \xi^2(\Lambda^{1+\epsilon})}{k\big(\Lambda^{1+\epsilon} \xi(\Lambda^{1+\epsilon})\big)}$$

in the range of n at least $\Lambda^{1+\epsilon}$ and for any Λ large enough. To see that this is less than $g_{[0,n)}/k(\Lambda)$, note that, considering the index of regular varition, the inequality

$$\frac{m(\Lambda^{1+\epsilon}) \xi^2(\Lambda^{1+\epsilon})}{k\big(\Lambda^{1+\epsilon} \xi(\Lambda^{1+\epsilon})\big)} \leqslant \frac{1}{k(\Lambda)}$$

holds since

$$(1+\epsilon)\Big(\beta - 1 + \frac{1}{\alpha}\Big) < -1 + \frac{1}{\alpha}$$

whenever

$$\beta < \frac{\epsilon}{1+\epsilon} \frac{\alpha - 1}{\alpha} .$$

Step 2. Using (3.2.4),

$$\mathrm{E} T_n^+ \sim \frac{\alpha}{\alpha - 1} g_{[0,n)} \frac{m_n^{1-1/\alpha}}{k(n)} \tag{3.2.7}$$

as n tends to infinity. Potter's bound to compare $k(n)$ and $k(n\xi_n)$ shows that (3.2.7) is less than (3.2.6), and therefore less than $g_{[0,n)}/k(\Lambda)$ in our range of n and Λ of interest. ∎

Combining Propositions 3.1.1 and 3.2.1, we see that, ignoring for the time being the lower tail of F, Theorem 2.2 will be proved if we show that for some ϵ positive,

$$\lim_{T \to \infty} \limsup_{\Lambda \to \infty} \mathrm{P}\{\, \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, |T_n^+ - \mathrm{E} T_n^+| \geqslant T g_{[0,n)}/k(\Lambda) \,\} = 0 . \tag{3.2.8}$$

As can be seen in the remainder of the proof, the fact that the innovations are kept in T_n^+ if they exceed a threshold b_n in which m_n depends on n creates some complications. So, it is better to backtrack from (3.2.8), and instead, still using Propositions 3.1.1 and 3.2.1, to argue that, still ignoring the problem of the lower tail

of F for the time being, Theorem 2.2 can be proved by showing that for any positive ϵ,

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\{\, \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, |S_n| \geqslant T g_{[0,n)}/k(\Lambda) \,\} = 0\,. \tag{3.2.9}$$

In the next step we show that we can now consider only the extreme values larger than some $b_{n,\Lambda}$ calculated with a sequence m_Λ instead of m_n.

3.3. Step 3. We now concentrate on the range of n between Λ and $\Lambda^{1+\epsilon}$. Imitating the notation used in step 1, let

$$b_{n,\Lambda} = F^{\leftarrow}\Bigl(1 - \frac{m_\Lambda}{n}\Bigr)$$

and similarly for $(a_{n,\Lambda})$. We set

$$\sigma_{n,\Lambda} = \mathrm{Var}\bigl(X \mathbb{1}_{[a_{n,\Lambda}, b_{n,\Lambda}]}(X)\bigr)\,.$$

Using the same notation as in step 1, but for a slightly different quantity — note indeed that we substitute $a_{n,\Lambda}$ and $b_{n,\Lambda}$ for a_n and b_n — consider the standardized middle innovations,

$$Z_{i,n} = \frac{X_i \mathbb{1}_{[a_{n,\Lambda}, b_{n,\Lambda}]}(X_i) - \mathrm{E} X \mathbb{1}_{[a_{n,\Lambda}, b_{n,\Lambda}]}(X)}{\sigma_{n,\Lambda}}\,, \qquad 1 \leqslant i \leqslant n\,.$$

Again, with a slight change of notation compared to step 1, the corresponding middle part of S_n is then

$$M_n = \sigma_{n,\Lambda} \sum_{0\leqslant i<n} g_i Z_{n-i,n}\,.$$

As in step 1, this middle part can be neglected in our problem.

Proposition 3.3.1. *For any β positive and less than 1, for any positive ϵ and T,*

$$\lim_{\Lambda\to\infty} \mathrm{P}\{\, \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, M_n > T g_{[0,n)}/k(\Lambda) \,\} = 0\,.$$

Proof. The same estimates as in Proposition 3.1.1 give the analogue of (3.1.1), namely, that for any positive integer r,

$$\mathrm{P}\{\, M_n > T g_{[0,n)}/k(\Lambda) \,\} \sim \frac{c}{T^r} m_\Lambda^{-r/2} \Bigl(\frac{k(\Lambda)}{k(n/m_\Lambda)}\Bigr)^r \tag{3.3.1}$$

as Λ tends to infinity and uniformly in n in $(\Lambda, \Lambda^{1+\epsilon})$. Let η be a positive real number. Potter's bound implies that uniformly in the range n in $(\Lambda, \Lambda^{1+\epsilon})$, as Λ tends to infinity,

$$k\Big(\frac{n}{m_\Lambda}\Big) \gtrsim m_\Lambda^{(1/\alpha)-1-\eta} k(n)\,.$$

If follows that (3.3.1) is of order at most

$$\frac{c}{T^r} m_\Lambda^{r((1/2)-(1/\alpha)+\eta)}\,.$$

Applying Bonferroni's inequality, we obtain

$$\begin{aligned} \mathrm{P}\{\, \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, M_n > T g_{[0,n)}/k(\Lambda) \,\} \qquad & \\ \leqslant \frac{c}{T^r} m_\Lambda^{r((1/2)-(1/\alpha)+\eta)} \Lambda^{1+\epsilon}\,. & \qquad (3.3.2) \end{aligned}$$

This bound is regularly varying in Λ of index $\beta r\big((1/2) - (1/\alpha) + \eta\big) + 1 + \epsilon$. This index is negative whenever η is small enough and r is large enough. ∎

3.4. Step 4. We consider the contribution of the extreme innovations to S_n. We consider $b_{n,\Lambda} = F^{\leftarrow}(1 - m_\Lambda/n)$ as in the previous subsection. With again a slight change of notation compared to step 2, we set

$$T_n^+ = \sum_{0 \leqslant i < n} g_i X_{n-i} \mathbb{1}_{(b_{n,\Lambda},\infty)}(X_{n-i})\,.$$

Paralleling what we did in step 2, we seek to prove the following proposition.

Proposition 3.4.1. *Let ϵ be a positive real number. If β is small enough, then for any positive T,*

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\{\, \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, |T_n^+ - \mathrm{E}T_n^+| \geqslant T g_{[0,n)}/k(\Lambda) \,\} = 0\,.$$

The proof of this proposition is far more difficult than that of Proposition 3.2.1. It first requires several approximations of T_n^+. Our goal at the end of these approximations amounts to be able to replace $T_n^+ - \mathrm{E}T_n^+$ by about $L_0^{(\gamma-1)}(n)/k(n)$.

Given a positive ϵ as in Proposition 3.4.1, we introduce for notational simplicity

$$N = \Lambda^{1+2\epsilon}\,.$$

The exponent could as well be taken to be $1+\epsilon$ but adding an extra ϵ will slightly simplify some of our arguments.

Following the construction in Barbe and McCormick (2010), let $X_{1,N} \leqslant X_{2,N} \leqslant \ldots \leqslant X_{N,N}$ be the order statistics of the innovations $(X_i)_{1\leqslant i\leqslant N}$. Let τ be the random permutation of $\{\,1,2,\ldots,N\,\}$ such that

$$X_{\tau(i)} = X_{N-i+1,N}\,.$$

We set g_i to be 0 if i is negative. For any n positive at most N, the equality

$$T_n^+ = \sum_{1\leqslant i\leqslant N} g_{n-\tau(i)} X_{N-i+1,N} \mathbb{1}_{(b_{n,\Lambda},\infty)}(X_{N-i+1,N}) \mathbb{1}\{\,\tau(i)\leqslant n\,\}$$

holds. Let $(V_i)_{1\leqslant i\leqslant N}$ be a sequence of independent random variables uniformly distributed over $[\,0,1\,]$, independent of $(X_{i,N})_{1\leqslant i\leqslant N}$. Let G_N be their empirical distribution function,

$$G_N(x) = N^{-1}\sum_{1\leqslant i\leqslant N} \mathbb{1}\{\,V_i\leqslant x\,\}\,.$$

Without any loss of generality, even if F is not continuous, we assume that $\tau(i) = NG_N(V_i)$, giving

$$\begin{aligned} T_n^+ = \sum_{1\leqslant i\leqslant N} & g_{n-NG_N(V_i)} X_{N-i+1,N} \\ & \qquad \mathbb{1}_{(b_{n,\Lambda},\infty)}(X_{N-i+1,N}) \mathbb{1}\{\,NG_N(V_i)\leqslant n\,\}\,. \end{aligned}$$

Let $(\omega_i)_{i\geqslant 1}$ be a sequence of independent random variables having a standard exponential distribution. For any positive integer i we define the partial sum $W_i = \omega_1+\cdots+\omega_i$. Since $(W_i/W_{N+1})_{1\leqslant i\leqslant N}$ has the same distribution as the order statistics of N independent uniform random variables (see Shorack and Wellner, 1986, chapter 8, §2),

$$(X_{N-i+1,N}) \stackrel{\mathrm{d}}{=} \Bigl(F^{\leftarrow}\Bigl(1-\frac{W_i}{W_{N+1}}\Bigr)\Bigr)_{1\leqslant i\leqslant N}\,.$$

Since we use the càglàd version of the quantile function, the inequality $F^{\leftarrow}(u) > b_{n,\Lambda}$ is equivalent to $u < m_\Lambda/n$. Therefore, introducing the random set

$$\mathcal{R}_{1,n,N} = \Big\{\, i \, : \, \frac{W_i}{W_{N+1}} \leqslant \frac{m_\Lambda}{n} \, ; \, G_N(V_i) \leqslant \frac{n}{N} \,\Big\}$$

and the random variable

$$T^+_{1,n,N} = \sum_{i\in\mathcal{R}_{1,n,N}} g_{n-NG_N(V_i)} F^{\leftarrow}\Big(1 - \frac{W_i}{W_{N+1}}\Big) \, , \qquad (3.4.1)$$

we have $(T^+_n)_{1\leqslant n\leqslant N} \stackrel{\mathrm{d}}{=} (T^+_{1,n,N})_{1\leqslant n\leqslant N}$.

Remark. The main reason we proved Propositions 3.1.1 and 3.2.1 is that once we can reduce n to be bounded from above by some given quantity — in our case $\Lambda^{1+\epsilon}$ — we can use the representation of the innovations with (W_i) and obtain (3.4.1).

Let $\mu^+_{n,\Lambda}$ be $\mathrm{E}X\mathbb{1}_{(b_{n,\Lambda},\infty)}(X)$, so that $\mathrm{E}T^+_n = g_{[0,n)}\mu^+_{n,\Lambda}$. Our discussion shows that in order to prove Proposition 3.4.1 it suffices to prove that

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\, \exists n \in (\Lambda, \Lambda^{2+\epsilon}) \, : \, |T^+_{1,n,N} - g_{[0,n)}\mu^+_{n,\Lambda}| > T\frac{g_{[0,n)}}{k(\Lambda)} \,\Big\} = 0 \, . \qquad (3.4.2)$$

This will be achieved by approximating $T^+_{1,n,N}$ by a much simpler process. While the main approximation scheme follows that in Barbe and McCormick (2010), a main difference lies in the sequential nature of the event involved in (3.4.2); indeed, contrary to Barbe and McCormick (2010) we need more than an approximation of $T^+_{1,n,N}$ valid for a range of n of comparable order, but for n between the different orders Λ and $\Lambda^{1+\epsilon}$, that is over a much larger range. To make this approximation, it is essential to have some understanding of the set $\mathcal{R}_{1,n,N}$.

The point process $\sum_{i\geqslant 1}\delta_{(W_i,V_i)}$ is a Poisson process of intensity the Lebesgue measure on $[\,0,\infty)\times[\,0,1\,]$. Viewing this point process in a (w,v)-plane, we can think of $\mathcal{R}_{1,n,N}$ as a region in this plane given by

$$\{\,(w,v) \, : \, w \leqslant W_{N+1}m_\Lambda/n \, ; \, G_N(v) \leqslant n/N \,\} \, .$$

So we will write about the 'set' $\mathcal{R}_{1,n,N}$ when we think of it as a subset of the integer and about the 'region' $\mathcal{R}_{1,n,N}$ when we view it as a set of points in the (w,v)-plane. The key fact to understand, which we will formalize in our next result, is that, considering $\mathcal{R}_{1,n,N}$, if n is small the inequality $G_N(V_i) \leqslant n/N$ will select few points and because V_i will be quite small, we will wait for a long time, that is we will need to have i large, in order to hit this V_i; but for large i, the sum W_i will be large and so W_i/W_{N+1} will not be less than m_Λ/n. If n is large, then the inequality $W_i/W_{N+1} \leqslant m_\Lambda/n$ forces i to be quite small, and so, regardless of n, we should have very few points in $\mathcal{R}_{1,n,N}$. A different argument, less informative in our context though, is that since we retained the innovations exceeding $F^{\leftarrow}(1-m_\Lambda/n)$ in $T^+_{1,n,N}$, there should be about m_Λ of those contributing to $T^+_{1,n,N}$. Thus, the cardinality of $\mathcal{R}_{1,n,N}$ should be about m_Λ.

Still considering $\mathcal{R}_{1,n,N}$, if we replace W_{N+1} by its near expected value N and approximate G_N by its limit, the identity function on $[\,0,1\,]$, we should have

$$V_i \leqslant \frac{n}{N} \qquad \text{and} \qquad W_i \leqslant m_\Lambda \frac{N}{n}\,.$$

These two inequalities force (V_i, W_i) to belong to the much simpler region $\{\,(v,w) \,:\, v \leqslant m_\Lambda/w\,\}$, which is the subgraph of an hyperbola. The replacement of G_N by its limit cannot be done at this early stage and we will have to settle for less, bounding G_N by a multiple of its limit. This leads us to introduce, for any positive c, the region

$$\mathcal{R}_{\Lambda,c} = \Big\{\,(v,w) \,:\, v \leqslant c\frac{m_\Lambda}{w}\;;\; w \leqslant 2N\,\Big\}\,.$$

Our next result asserts that provided c is large enough, it is very likely for the regions $\mathcal{R}_{1,n,N}$ to be included in $\mathcal{R}_{\Lambda,c}$, and that the minimum of the set $\mathcal{R}_{1,n,N}$ is very likely to be at least $N/n\xi_N$.

Lemma 3.4.2. *For any positive η there exists a real number c such that for any Λ large enough,*

$$\mathrm{P}\Big(\bigcap\nolimits_{\Lambda\leqslant n\leqslant\Lambda^{1+\epsilon}}\{\,\mathcal{R}_{1,n,N} \subset \mathcal{R}_{\Lambda,c}\,\}\Big) \geqslant 1-\eta\,.$$

Moreover, viewing $\mathcal{R}_{1,n,N}$ as a set of integers,

$$\lim_{\Lambda\to\infty} \mathrm{P}\Big(\bigcap\nolimits_{\Lambda\leqslant n\leqslant\Lambda^{1+\epsilon}}\Big\{\min \mathcal{R}_{1,n,N} \geqslant \frac{N}{n\xi_N}\,\Big\}\Big) = 1\,.$$

Proof. Let η be an arbitrary positive real number. Inequality (2) in Shorack and Wellner (1978) (see also Shorack and Wellner, 1986, chapter 10, §3, inequality 1, p. 415) and the strong law of large numbers ensure the existence of c such that the event

$$\Omega_N = \bigcap_{1\leqslant i\leqslant N} \Big\{ V_{i,N} \leqslant c\frac{i}{N} \Big\} \cap \{ W_{N+1} \leqslant 2N \}$$

has probability at least $1-\eta$ whenever N is large enough. On Ω_N, if i belongs to $\mathcal{R}_{1,n,N}$,

$$V_i \leqslant V_{n,N} \leqslant c\frac{n}{N} \leqslant cm_\Lambda \frac{W_{N+1}}{W_i}\frac{1}{N} \leqslant 2c\frac{m_\Lambda}{W_i} \, ,$$

the first equality coming from the condition $G_N(V_i) \leqslant n/N$, the second inequality from being in Ω_N, the third from the condition $W_i/W_{N+1} \leqslant m_\Lambda/n$, and the last from being in Ω_N. Thus, up to replacing c by twice as much, $\mathcal{R}_{1,n,N} \subset \mathcal{R}_{\Lambda,c}$ on Ω_N, proving the first assertion.

Given (3.2.1), Geffroy (1958/1959) or Kiefer's (1972) Theorem 1 imply that for c small enough the event

$$\Omega = \bigcap_{i\geqslant 1}\{ V_{1,i} \geqslant c/i\xi_i \}$$

has probability at least $1-\eta$. If i belongs to $\mathcal{R}_{1,n,N}$ and Ω_N occurs, $V_i \leqslant cn/N$, and, in particular, $V_{1,i} \leqslant cn/N$. Therefore, on $\Omega\cap\Omega_N$ we must have $1/(i\xi_i) \leqslant cn/N$, that is, $i\xi_i \geqslant cN/n$. In particular, $i \geqslant cN/(n\xi_i)$. However, if i is in $\mathcal{R}_{1,n,N}$, then i is at most N, and since (ξ_n) is ultimately monotone, we obtain that i is at least $cN/\big(n\xi_N\big)$. ∎

As a consequence of Lemma 3.4.2, we can show that the region $\mathcal{R}_{1,n,N}$ cannot contain too many points if β is small.

Lemma 3.4.3. $\max_{\Lambda\leqslant n\leqslant\Lambda^{2+\epsilon}} \sharp\mathcal{R}_{1,n,N} = O_P(m_\Lambda \log N)$.

Proof. Since V_i is uniform over $[\,0,1\,]$ the region $\mathcal{R}_{\Lambda,c}$ can be restricted to

$$\{ (v,w) \,:\, v \leqslant (cm_\Lambda/w)\wedge 1 \,;\, w \leqslant 2N \} \, .$$

The Lebesgue measure of this region is of order $cm_\Lambda \log(2N)$. The result follows since $\sum_{i\geqslant 1} \delta_{(V_i,W_i)}$ is a homogenous Poisson process with mean intensity 1 and Lemma 3.4.2 holds. ∎

We will use the following lemma, which we state now for convenience.

Lemma 3.4.4. *Let $(\Pi_{i,n})_{1\leqslant i\leqslant n}$ be some Poisson random variables, possibly dependent, having respective means $(\lambda_{i,n})_{1\leqslant i\leqslant n}$, such that for some ϵ positive, $\max_{1\leqslant i\leqslant n} \lambda_{i,n} = o(n^{-\epsilon})$. Then, for $k \geqslant 2/\epsilon$,*

$$\lim_{n\to\infty} \mathrm{P}\{\, \max_{1\leqslant i\leqslant n} \Pi_{i,n} \geqslant k \,\} = 0\,.$$

Proof. Chernoff's inequality yields for any positive k,

$$\mathrm{P}\{\, \Pi_{i,n} \geqslant k \,\} \leqslant \exp(-k\log k + k\log \lambda_{i,n} + k - \lambda_{i,n})\,.$$

Given the assumption on $(\lambda_{i,n})$ this upper bound is, for any n large enough, at most $\exp(-k\log k - \epsilon k \log n + k)$. The result follows from Bonferroni's inequality. ∎

Since we will make repeated use of the following simple argument or obvious variants of it, we state it as a lemma.

Lemma 3.4.5. *Let (ϵ_n) be a bounded sequence of positive real numbers. There exists a positive T such that for any n at least Λ and any Λ large enough,*

$$F^{\leftarrow}(1-1/n) g_n \epsilon_n \leqslant T g_{[0,n)}/k(\Lambda)\,.$$

Proof. Given (3.1), the inequality amounts to $\epsilon_n < ck(n)/k(\Lambda)$. Since the function k is regularly varying of positive index,

$$\lim_{\Lambda\to\infty} \inf_{n\geqslant\Lambda} k(n)/k(\Lambda) = 1\,,$$

and the result follows. ∎

Having made these observations on $\mathcal{R}_{1,n,N}$, we can start a long string of approximations of $T^+_{1,n,N}$. Referring to (3.4.1), we first replace $F^{\leftarrow}(1-W_i/W_{N+1})$ by $F^{\leftarrow}(1-1/n)(nW_i/W_{N+1})^{-1/\alpha}$. Define

$$T^+_{2,n,N} = F^{\leftarrow}\Bigl(1-\frac{1}{n}\Bigr)\Bigl(\frac{W_{N+1}}{n}\Bigr)^{1/\alpha} \sum_{i\in\mathcal{R}_{1,n,N}} g_{n-NG_N(V_i)} W_i^{-1/\alpha}\,.$$

Our next lemma shows that we can replace $T^+_{1,n,N}$ by $T^+_{2,n,N}$ in order to prove (3.4.2). Recall that β refers to the index of regular variation of (m_Λ) as a function of Λ, and that, except if specified otherwise, we allow it to vanish under (3.2.2) and (3.2.3).

Lemma 3.4.6. *If β is less than κ, then*

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\, \exists n\in(\Lambda,\Lambda^{1+\epsilon}) \,:\, |T^+_{1,n,N}-T^+_{2,n,N}| > T\frac{g_{[0,n)}}{k(\Lambda)}\,\Big\} = 0\,.$$

Proof. Let i be an integer in $\mathcal{R}_{1,n,N}$. We have $nW_i/W_{N+1}\leqslant m_\Lambda$. Lemma 3.4.2 implies that except on a set whose probability can be made arbitrary small by taking N large enough, i is at least $N/n\xi_N$. Hence, the strong law of large numbers yields that W_i is at least $N/2n\xi_N$ provided N is large enough. Then, nW_i/W_{N+1} is at least $1/4\xi_N$. Consequently, (3.2.2) and (3.2.3) yield

$$\begin{aligned} &F^{\leftarrow}\Big(1-\frac{W_i}{W_{N+1}}\Big)\\ &\qquad = F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{nW_i}{W_{N+1}}\Big)^{-1/\alpha} + F^{\leftarrow}\Big(1-\frac{1}{n}\Big)O_P(m_\Lambda^{-\rho})\,, \end{aligned}$$

the $O_P(m_\Lambda^{-\rho})$ being uniform in i in $\mathcal{R}_{1,n,N}$. Since (g_n) is asymptotically equivalent to a nondecreasing sequence and Lemma 3.4.3 holds, we obtain

$$\begin{aligned} T^+_{1,n,N} &= T^+_{2,n,N} + F^{\leftarrow}\Big(1-\frac{1}{n}\Big)g_n m_\Lambda^{-\rho}\sharp\mathcal{R}_{1,n,N}O_P(1)\\ &= T^+_{2,n,N} + F^{\leftarrow}\Big(1-\frac{1}{n}\Big)g_n o_P(1)\,, \end{aligned}$$

the o_P-term being uniform in n between Λ and $\Lambda^{1+\epsilon}$. A variation on Lemma 3.4.5 implies the result. ■

In $T^+_{2,n,N}$, we replace W_{N+1} by N, setting

$$T^+_{3,n,N} = F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha}\sum_{i\in\mathcal{R}_{1,n,N}} g_{n-NG_N(V_i)}W_i^{-1/\alpha}\,.$$

Lemma 3.4.7. *If β is less than $1/2$,*

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\, \exists n\in(\Lambda,\Lambda^{1+\epsilon}) \,:\, |T^+_{2,n,N}-T^+_{3,n,N}| > T\frac{g_{[0,n)}}{k(\Lambda)}\,\Big\} = 0\,.$$

Proof. Taylor's formula, the central limit theorem and the strong law of large numbers yield

$$W_{N+1}^{1/\alpha} - N^{1/\alpha} = N^{(1/\alpha)-(1/2)} O_P(1)$$

as N tends to infinity. Lemma 3.4.2 and the strong law of large numbers imply that if i is in $\mathcal{R}_{1,n,N}$, then $W_i^{-1/\alpha} \leqslant 2\big(n\xi_N/N\big)^{1/\alpha}$ whenever N is large enough. Therefore, using Lemma 3.4.3,

$$\begin{aligned}
&|T^+_{2,n,N} - T^+_{3,n,N}| \\
&\qquad \leqslant F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} N^{-1/2} g_n \sharp\mathcal{R}_{1,n,N}\Big(\frac{n\xi_N}{N}\Big)^{1/\alpha} O_P(1) \\
&\qquad = F^{\leftarrow}\Big(1-\frac{1}{n}\Big) g_n \frac{m_\Lambda \xi^{1/\alpha}(N)}{\sqrt{N}} \log N O_P(1)\,,
\end{aligned}$$

uniformly in n between Λ and $\Lambda^{1+\epsilon}$. The result follows by a simple adaptation of Lemma 3.4.5. ∎

Seeking to replace $\mathcal{R}_{1,n,N}$ by the slightly simpler set

$$\mathcal{R}_{2,n,N} = \Big\{\, i \,:\, W_i \leqslant m_\Lambda \frac{N}{n}\,;\, G_N(V_i) \leqslant \frac{n}{N} \,\Big\}\,,$$

we set

$$T^+_{4,n,N} = F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} \sum_{i\in\mathcal{R}_{2,n,N}} g_{n-NG_N(V_i)} W_i^{-1/\alpha}\,.$$

Lemma 3.4.8. *For any β less than $(1/2)-\epsilon$,*

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\, \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, |T^+_{3,n,N} - T^+_{4,n,N}| \geqslant T \frac{g_{[0,n)}}{k(\Lambda)} \,\Big\} = 0\,.$$

Proof. If i belongs to the symmetric difference $\mathcal{R}_{1,n,N} \triangle \mathcal{R}_{2,n,N}$, then $G_N(V_i) \leqslant n/N$, and either

$$m_\Lambda \frac{W_{N+1}}{n} \leqslant W_i \leqslant m_\Lambda \frac{N}{n} \qquad \text{or} \qquad m_\Lambda \frac{N}{n} \leqslant W_i \leqslant m_\Lambda \frac{W_{N+1}}{n}\,.$$

Thus, W_i lies in a region of width

$$\frac{m_\Lambda}{n}|W_{N+1}-N| = \frac{m_\Lambda}{n}\sqrt{N}O_P(1)$$

with an endpoint given by $m_\Lambda N/n$. In particular, $i \sim m_\Lambda N/n$ and (V_i, W_i) lies in a region of area of order at most

$$\frac{m_\Lambda}{n}\sqrt{N}O_P(1) = m_\Lambda \Lambda^{-(1/2)+\epsilon}O_P(1)\,,$$

the $O_P(1)$-term being uniform in $\Lambda \leqslant n \leqslant \Lambda^{1+\epsilon}$. In particular, taking β less than $(1/2)-\epsilon$, this area tends to 0 at algebraic rate. Applying Lemma 3.4.4, there exists a positive k such that

$$\lim_{\Lambda\to\infty} \mathrm{P}\big\{ \max_{\Lambda\leqslant n\leqslant\Lambda^{1+\epsilon}} \sharp(\mathcal{R}_{1,n,N} \triangle \mathcal{R}_{2,n,N}) \geqslant k \,\big\} = 0\,.$$

Using again that all i in $\mathcal{R}_{1,n,N} \triangle \mathcal{R}_{2,n,N}$ are asymptotically equivalent to $m_\Lambda N/n$ and using also the strong law of large numbers, we obtain

$$\begin{aligned}
&|T^+_{3,n,N} - T^+_{4,n,N}| \\
&\leqslant F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n \max_{i\in\mathcal{R}_{1,n,N}\triangle\mathcal{R}_{2,n,N}} W_i^{-1/\alpha}\sharp(\mathcal{R}_{1,n,N} \triangle \mathcal{R}_{2,n,N}) \\
&\leqslant F^{\leftarrow}\Big(1-\frac{1}{n}\Big) g_n m_\Lambda^{-1/\alpha} O_P(1)\,.
\end{aligned}$$

The result then follows from Lemma 3.4.5. ∎

Considering $T^+_{4,n,N}$, we seek to replace $g_{n-NG_N(V_i)}$ by $g_{\lfloor n-NV_i\rfloor}$. For simplicity, we will write g_{n-NV_i} for the latter. Therefore, we define

$$T^+_{5,n,N} = F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} \sum_{i\in\mathcal{R}_{2,n,N}} g_{n-NV_i} W_i^{-1/\alpha}\,.$$

Lemma 3.4.9. *For any β less than $\big((\gamma-1)\wedge 1\big)/4$,*

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty} \mathrm{P}\Big\{ \exists n \in (\Lambda,\Lambda^{1+\epsilon}) \,:\, |T^+_{4,n,N} - T^+_{5,n,N}| > T\frac{g_{[0,n)}}{k(\Lambda)} \Big\} = 0\,.$$

Proof. Let ϵ_1 and η be two positive real numbers. Since the function $\big(\mathrm{Id}(1-\mathrm{Id})\big)^{(1/2)-\eta}$ is a Chibisov-O'Reilly function (see for instance Csörgő, Csörgő, Horvàth and Mason, 1986, Theorem 4.2.3),

$$\max_{1\leqslant i\leqslant N} \frac{\sqrt{N}|G_N(V_i)-V_i|}{\big(V_i(1-V_i)\big)^{(1/2)-\eta}} = O_P(1)$$

as N tends to infinity. If i belongs to $\mathcal{R}_{2,n,N}$, then $G_N(V_i)\leqslant n/N$ and, using Shorack and Wellner's (1978, inequality (2)) linear bounds, $V_i\leqslant cn/N$ with probability at least $1-\epsilon_1$ provided c is large enough. Thus,

$$\max_{i\in\mathcal{R}_{2,n,N}} N|G_N(V_i)-V_i| \leqslant \sqrt{N}\Big(\frac{n}{N}\Big)^{(1/2)-\eta} O_P(1) \tag{3.4.3}$$

where the $O_P(1)$-term is uniform in n between Λ and $\Lambda^{2+\epsilon}$.

For any integer r let

$$\Omega_n(r) = \max_{1\leqslant i\leqslant n-r} |g_{i+r}-g_i|\,.$$

Inequality (3.4.3) implies that with probability at least $1-\epsilon_1$ provided c is large enough,

$$\max_{i\in\mathcal{R}_{2,n,N}} |g_{n-NG_N(V_i)}-g_{n-NV_i}| \leqslant \max_{0\leqslant r\leqslant cn^{(1/2)-\eta}N^\eta} \Omega_n(r)\,. \tag{3.4.4}$$

Recall that (2.2) holds. According to whether γ is at least 2 or not, Lemmas 5.6 and 5.8 in Barbe and McCormick (2010) imply that whenever θ is a positive real number less than $(\gamma-1)\wedge 1$, the right hand side of (3.4.4) is at most $cg_n(n^{-(1/2)-\eta}N^\eta)^\theta$. Since n is at least Λ in our range of interest, the right hand side of (3.4.4) is of order g_n times Λ at the power $-\theta/2+O(\eta)$. Thus, if η is small and n and N are large enough, enough,

$$\begin{aligned}
&|T^+_{4,n,N}-T^+_{5,n,N}|\\
&\qquad\leqslant F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_nN^{-\theta/4}\sum_{i\in\mathcal{R}_{2,n,N}} W_i^{-1/\alpha}O_P(1)\\
&\qquad\leqslant F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_nN^{-\theta/4}\max_{i\in\mathcal{R}_{2,n,N}} W_i^{-1/\alpha}\sharp\mathcal{R}_{2,n,N}O_P(1)\,.
\end{aligned}$$

Lemma 3.4.2 with $\mathcal{R}_{2,n,N}$ substituted for $\mathcal{R}_{1,n,N}$, the strong law of large numbers applied to the sums $(W_i)_{i\geqslant 1}$ and Lemma 3.4.3 with

$\mathcal{R}_{2,n,N}$ substituted for $\mathcal{R}_{1,n,N}$ shows that the above upper bound is, in probability, of order

$$\begin{aligned} F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n N^{-\theta/4}\Big(\frac{n\xi_N}{N}\Big)^{1/\alpha} m_\Lambda \log N \\ = F^{\leftarrow}\Big(1-\frac{1}{n}\Big) g_n N^{-\theta/4}\xi_N^{1/\alpha} m_\Lambda \log N\,. \end{aligned}$$

Thus, if β is less than $\theta/4$, Lemma 3.4.5 implies the result. ∎

Next, using the regular variation of the sequence (g_n), we would like to replace g_{n-NV_i} in $T^+_{5,n,N}$ by $g_n(1-NV_i/n)^{\gamma-1}$. This leads us to define

$$T^+_{6,n,N} = F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n \sum_{i\in\mathcal{R}_{2,n,N}} \Big(1-\frac{N}{n}V_i\Big)_+^{\gamma-1} W_i^{-1/\alpha}\,.$$

With regard to the next lemma, recall that δ was introduced in (2.3).

Lemma 3.4.10. *If β is less than δ, then*

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\,\exists n\in(\Lambda,\Lambda^{1+\epsilon})\ :\ |T^+_{5,n,N}-T^+_{6,n,N}|>T\frac{g_{[0,n)}}{k(\Lambda)}\,\Big\}=0\,.$$

Proof. To approximate $T^+_{5,n,N}$ by $T^+_{6,n,N}$, we need to rely on assumption (2.3). With respect to this assumption, we see that $\mathcal{R}_{2,n,N}$ contains 'good' points, for which

$$n^{-\delta}\leqslant\frac{n-NV_i}{n}\leqslant 1\,,$$

guaranteeing that with (2.3) we can substitute $g_n(1-NV_i/n)^{\gamma-1}$ for g_{n-NV_i}, and 'bad' points, for which either

$$0\leqslant\frac{n-NV_i}{n}\leqslant n^{-\delta}\qquad\text{or}\qquad V_i>n/N\,.$$

We call $\mathcal{B}_{1,n,N}$ the set of all bad points for which $0\leqslant n-NV_i\leqslant n^{1-\delta}$ and $\mathcal{B}_{2,n,N}$ the set of those for which $V_i>n/N$.

Let i be in $\mathcal{B}_{1,n,N}$. Since it belongs to $\mathcal{R}_{2,n,N}$, Lemma 3.4.2 shows that with probability arbitrarily close to 1 provided N is large

enough, $i \geqslant N/n\xi_N$. Therefore, since i is in $\mathcal{R}_{2,n,N}$, with probability arbitrarily close to 1 provided N is large enough,

$$\frac{N}{2n\xi_N} \leqslant W_i \leqslant m_\Lambda \frac{N}{n}\,.$$

And since i is a bad point in $\mathcal{B}_{1,n,N}$,

$$\frac{n-n^{1-\delta}}{N} \leqslant V_i \leqslant \frac{n}{N}\,.$$

When thinking of $\mathcal{B}_{1,n,N}$ as a region as we did with $\mathcal{R}_{1,n,N}$, we thus have with high probability,

$$\mathcal{B}_{1,n,N} \subset \Big\{\,(v,w)\,:\,\frac{n-n^{1-\delta}}{N} \leqslant v \leqslant \frac{n}{N}\,;\,\frac{N}{2n\xi_N} \leqslant w \leqslant m_\Lambda \frac{N}{n}\,\Big\}\,.$$

The area of this upper bound is of order $(n^{1-\delta}/N)(m_\Lambda N/n) = m_\Lambda/n^\delta$ and tends to 0 at an algebraic rate in Λ provided β is less than δ. Therefore, Lemma 3.4.4 implies

$$\max_{\Lambda \leqslant n \leqslant \Lambda^{1+\epsilon}} \sharp\mathcal{B}_{1,n,N} = O_P(1)$$

as Λ tends to infinity.

Considering the bad points in $T^+_{5,n,N}$, we have, since (g_n) is equivalent to a nondecreasing sequence,

$$\begin{aligned}
\Big| F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} &\sum_{i\in\mathcal{B}_{1,n,N}} g_{n-NV_i} W_i^{-1/\alpha} \Big| \\
&\leqslant F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_{n^{1-\delta}} \max_{i\in\mathcal{R}_{2,n,N}} W_i^{-1/\alpha} \sharp\mathcal{B}_{1,n,N} \qquad (3.4.5)\\
&= F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n n^{-\delta(\gamma-1)} \Big(\frac{n\xi_N}{N}\Big)^{1/\alpha} O_P(1)\,.
\end{aligned}$$

This upper bound is of order $F^{\leftarrow}(1-1/n)g_n n^{-\delta(\gamma-1)}\xi^{1/\alpha}(N)$; since n is at least Λ and ξ is slowly varying, it is of order at most $F^{\leftarrow}(1-1/n)g_n n^{-\delta(\gamma-1)/2}$. Lemma 3.4.5 then implies

$$\begin{aligned}
\lim_{\Lambda\to\infty} \mathrm{P}\Big\{\,\exists n \in (\Lambda,\Lambda^{1+\epsilon})\,:\, F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha}& \\
\sum_{i\in\mathcal{B}_{1,n,N}} g_{n-NV_i} W_i^{-1/\alpha} > T g_{[0,n)}/k(\Lambda)\,\Big\} = 0\,.& \qquad (3.4.6)
\end{aligned}$$

Referring to $T^+_{6,n,N}$,

$$\begin{aligned} F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n \sum_{i\in\mathcal{B}_{1,n,N}} \Big(1-\frac{N}{n}V_i\Big)_+^{\gamma-1} W_i^{-1/\alpha} \\ \leqslant F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n n^{-\delta(\gamma-1)} \max_{i\in\mathcal{R}_{2,n,N}} W_i^{-1/\alpha} \sharp\mathcal{B}_{1,n,N}\,, \end{aligned}$$

which is the same bound as in (3.4.5). Therefore, the analogue of (3.4.6) holds when substituting $g_n(1-NV_i/n)_+^{\gamma-1}$ for g_{n-NV_i}.

Dealing with the bad points in $\mathcal{B}_{2,n,N}$ is easy because if $V_i > n/N$ then g_{n-NV_i} and $(1-NV_i/n)_+^{\gamma-1}$ vanish. So those points do not contribute to $T^+_{5,n,N}$ and $T^+_{6,n,N}$.

On the part of $T^+_{5,n,N}$ made by the good points, assumption (2.3) and Lemma 3.4.3 yield, on the range of n between Λ and $\Lambda^{1+\epsilon}$,

$$\begin{aligned} \sum_{i\in\mathcal{R}_{2,n,N}\setminus(\mathcal{B}_{1,n,N}\cup\mathcal{B}_{2,n,N})} \Big| g_{n-NV_i} - g_n\Big(1-\frac{NV_i}{n}\Big)^{\gamma-1}\Big| W_i^{-1/\alpha} \\ \leqslant n^{-\delta} g_n \max_{i\in\mathcal{R}_{2,n,N}} W_i^{-1/\alpha} \sharp\mathcal{R}_{2,n,N} \\ \leqslant \Lambda^{-\delta} g_n \Big(\frac{n\xi_N}{N}\Big)^{1/\alpha} m_\Lambda \log N O_P(1)\,. \end{aligned}$$

The bound obtained for the error in the approximation in $T^+_{5,n,N}$ for the good points is then

$$F^{\leftarrow}\Big(1-\frac{1}{n}\Big) g_n \xi_N^{1/\alpha} \Lambda^{-\delta} m_\Lambda \log N O_P(1)\,.$$

If β is less than δ then $\xi_N^{1/\alpha}\Lambda^{-\delta} m_\Lambda \log N$ tends to 0 as Λ tends to infinity and Lemma 3.4.5 yields the conclusion. ∎

We now replace $\mathcal{R}_{2,n,N}$ in $T^+_{6,n,N}$ by

$$\mathcal{R}_{3,n,N} = \Big\{\, i \,:\, W_i \leqslant m_\Lambda \frac{N}{n}\,,\, V_i \leqslant \frac{n}{N}\,\Big\}\,,$$

defining

$$T^+_{7,n,N} = F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n \sum_{i\in\mathcal{R}_{3,n,N}} \Big(1-\frac{N}{n}V_i\Big)_+^{\gamma-1} W_i^{-1/\alpha}\,.$$

Lemma 3.4.11. *If β is less than $(\gamma-1)/2$,*

$$\lim_{\Lambda\to\infty} \mathrm{P}\Big\{ \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, |T^+_{6,n,N} - T^+_{7,n,N}| > T\frac{g_{[0,n)}}{k(\Lambda)} \Big\} = 0\,.$$

Proof. If i belongs to $\mathcal{R}_{2,n,N} \triangle \mathcal{R}_{3,n,N}$, then either

$$V_i \leqslant \frac{n}{N} \leqslant G_N(V_i) \qquad \text{or} \qquad G_N(V_i) \leqslant \frac{n}{N} \leqslant V_i\,.$$

In the latter case, $(1 - NV_i/n)_+$ vanishes and so those points do not contribute to $T^+_{6,n,N}$ and $T^+_{7,n,N}$. In the former case, arguing as in the proof of Lemma 3.4.9,

$$\begin{aligned} V_i \leqslant \frac{n}{N} \leqslant G_N(V_i) &\leqslant V_i + \frac{V_i^{(1/2)-\eta}}{\sqrt{N}} O_P(1) \\ &\leqslant V_i + \frac{n^{(1/2)-\eta}}{N} N^\eta O_P(1)\,, \end{aligned}$$

the last inequality coming from the first one and the $O_P(1)$-term being uniform over i in $\mathcal{R}_{2,n,N} \triangle \mathcal{R}_{3,n,N}$ and n between Λ and $\Lambda^{1+\epsilon}$. In particular,

$$\Big|1 - \frac{N}{n} V_i\Big| \leqslant n^{-(1/2)-\eta} N^\eta O_P(1)\,.$$

Consequently, using lemma 3.4.3 with $\mathcal{R}_{3,n,N}$ substituted for $\mathcal{R}_{2,n,N}$,

$$\begin{aligned} &|T^+_{6,n,N} - T^+_{7,n,N}| \\ &\quad\leqslant F^\leftarrow\Big(1 - \frac{1}{n}\Big)\Big(\frac{N}{n}\Big)^{1/\alpha} g_n (n^{-(1/2)-\eta} N^\eta)^{\gamma-1} \Big(\frac{n}{N}\xi_N\Big)^{1/\alpha} \\ &\qquad\qquad \times \sharp(\mathcal{R}_{2,n,N} \triangle \mathcal{R}_{3,n,N}) O_P(1) \\ &\quad\leqslant F^\leftarrow\Big(1 - \frac{1}{n}\Big) g_n \xi_N^{2/\alpha} (n^{-(1/2)-\eta} N^\eta)^{\gamma-1} m_\Lambda \log N O_P(1)\,. \end{aligned}$$

Note that $n^{-(1/2)-\eta}N^\eta$ is at most $\Lambda^{-(1/2)-\eta+(1+2\epsilon)\eta}$. Provided η is small enough,

$$\Big(-\frac{1}{2} - \eta + (1+2\epsilon)\eta\Big)(\gamma-1) + \beta$$

is negative. We apply Lemma 3.4.5 to conclude. ∎

Having approximated all these $T^+_{i,n,N}$, we need to consider their expected values, a much easier task. This requires us to have an estimate on how close $g_{[0,n)}$ is to ng_n/γ and how large $\mu^+_{n,\Lambda}$ is. We establish those estimates in the next two lemmas.

Lemma 3.4.12. *If (2.3) holds, then, as n tends to infinity,*

$$\frac{\gamma g_{[0,n)}}{ng_n} = 1 + o(n^{-\delta})\,.$$

Proof. We write $\gamma g_{[0,n)}/ng_n$ as

$$\frac{\gamma}{n}\sum_{0\leqslant i<n^{1-\delta}}\frac{g_i}{g_n} + \frac{\gamma}{n}\sum_{n^{1-\delta}\leqslant i<n}\frac{g_i}{g_n}\,. \tag{3.4.7}$$

Since (g_n) is asymptotically equivalent to a monotone sequence, the first term in (3.4.7) is at most $O(1)n^{-\delta}g_{n^{1-\delta}}/g_n = o(n^{-\delta})$. Using (2.3), the second term is at most

$$\begin{aligned}
&\frac{\gamma}{n}\sum_{n^{1-\delta}<i\leqslant n}\Big(\Big(\frac{i}{n}\Big)^{\gamma-1} + o(n^{-\delta})\Big)\\
&\qquad\qquad = \frac{1}{n^\gamma}\Big(\int_{n^{1-\delta}}^{n}\gamma x^{\gamma-1}\,\mathrm{d}x + O(1)\Big) + o(n^{-\delta})\\
&\qquad\qquad = 1 + O(n^{-\gamma\delta}) + o(n^{-\delta})\,.
\end{aligned}$$

The result follows since γ exceeds 1. ∎

The proof of Lemma 3.2.2 shows that if β is small enough,

$$\limsup_{n\to\infty}\sup_{\Lambda\leqslant n\leqslant\Lambda^{1+\epsilon}}\Big|k(n)\mu^+_{n,\Lambda} - \frac{\alpha}{\alpha-1}m_\Lambda^{1-1/\alpha}\Big| < \infty\,. \tag{3.4.8}$$

Lemma 3.4.13. *If $\beta < \delta\alpha/(\alpha-1)$, there exists a positive T such that for any Λ large enough and any n in $(\Lambda,\Lambda^{1+\epsilon})$,*

$$|\mathrm{E}T^+_{1,n,N} - \mathrm{E}T^+_{7,n,N}| \leqslant T\frac{g_{[0,n)}}{k(\Lambda)}\,.$$

Proof. Lemmas 3.4.12 and (3.4.8) yield

$$\begin{aligned} \mathrm{E}T^+_{1,n,N} - \frac{ng_n}{\gamma}\mu^+_{n,\Lambda} &= ng_n\mu^+_{n,\Lambda} o(n^{-\delta}) \\ &= g_n F^{\leftarrow}\Big(1-\frac{1}{n}\Big) m_\Lambda^{1-1/\alpha} n^{-\delta} o(1)\,. \end{aligned} \tag{3.4.9}$$

The sequence $m_\Lambda^{1-1/\alpha}\Lambda^{-\delta}$ is regularly varying of index $\beta(1-1/\alpha)-\delta$, which is negative provided β is small enough. Thus, Lemma 3.4.5 shows that we can replace $\mathrm{E}T^+_{1,n,N}$ by $ng_n\mu^+_{n,\Lambda}/\gamma$ to prove the current lemma.

The calculation of $\mathrm{E}T^+_{7,n,N}$ can be done by using that W_i has a gamma distribution with parameter i, but an easier argument will show after our next lemma that

$$\mathrm{E}T^+_{7,n,N} = F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\frac{g_n}{\gamma}\frac{\alpha}{\alpha-1} m_\Lambda^{1-1/\alpha}\,. \tag{3.4.10}$$

Thus, it suffices to show that for any T large enough,

$$F^{\leftarrow}\Big(1-\frac{1}{n}\Big)\frac{g_n}{\gamma}\Big(k(n)\mu^+_{n,\Lambda} - \frac{\alpha}{\alpha-1}m_\Lambda^{1-1/\alpha}\Big) \leqslant Tg_{[0,n)}/k(\Lambda)\,.$$

This follows from (3.4.8) and Lemma 3.4.5. ■

Combining Lemmas 3.4.6 – 3.4.11 and 3.4.13, we see that in order to prove Proposition 3.4.1, it suffices to show that

$$\limsup_{\Lambda\to\infty} \mathrm{P}\{\,\exists n \in (\Lambda,\Lambda^{1+\epsilon}) \,:\, |T^+_{7,n,N} - \mathrm{E}T^+_{7,n,N}| \geqslant Tg_{[0,n)}/k(\Lambda)\,\} = 0\,. \tag{3.4.11}$$

Our next lemma will allow us to represent $T^+_{7,n,N}$ as an integral of a Poisson process over the quadrant $[\,0,\infty)^2$ and prove a valuable scaling property.

Lemma 3.4.14. *We can construct a Poisson process Π on $[\,0,\infty)^2$ with mean intensity the Lebesgue measure, such that the point process obtained by restricting Π to $[\,0,1\,]\times[\,0,\infty)$ coincides with $\sum_{i\geqslant 1}\delta_{(V_i,W_i)}$.*

Proof. Let N' be a homogenous and unit intensity Poisson random measure on $(1,\infty)\times[\,0,\infty)$, independent of the sequence (V_i,W_i). Define N as $N'+\sum_{i\geqslant 1}\delta_{(V_i,W_i)}$. ■

We then rewrite $T^+_{7,n,N}$ as

$$F^{\leftarrow}\Big(1-\frac{1}{n}\Big)g_n\Big(\frac{N}{n}\Big)^{(1/\alpha)+\gamma-1}\sum_{i\geqslant 1}\Big(\frac{n}{N}-V_i\Big)_+^{\gamma-1}W_i^{-1/\alpha}$$
$$\mathbb{1}\Big\{\frac{n}{N}W_i\leqslant m_\Lambda\Big\}\,.$$

In this expression, considering only the sum over i and thinking of n/N as a continuous variable t, we are led to introduce the process

$$\Upsilon_\Lambda(t)=\sum_{i\geqslant 1}(t-V_i)_+^{\gamma-1}W_i^{-1/\alpha}\mathbb{1}\{\,tW_i\leqslant m_\Lambda\,\}$$

indexed by t in $[\,0,1\,]$. Given Lemma 3.4.14, we can extend Υ_Λ to a process over the nonnegative half-line

$$\Upsilon_\Lambda(t)=\int(t-v)_+^{\gamma-1}w^{-1/\alpha}\mathbb{1}\{\,tw\leqslant m_\Lambda\,\}\,\mathrm{d}\Pi(v,w)\,.$$

We then have

$$T^+_{7,n,N}=F^{\leftarrow}\Big(1-\frac{1}{n}\Big)g_n\Big(\frac{N}{n}\Big)^{(1/\alpha)+\gamma-1}\Upsilon_\Lambda\Big(\frac{n}{N}\Big)\,. \tag{3.4.12}$$

In particular, $\mathrm{E}T^+_{7,n,N}$ is indeed given by (3.4.10) since the intensity of Π being the Lebesgue measure,

$$\begin{aligned}\mathrm{E}\Upsilon_\Lambda(t)&=\int(t-v)_+^{\gamma-1}w^{-1/\alpha}\mathbb{1}\{\,tw\leqslant m_\Lambda\,\}\,\mathrm{d}v\,\mathrm{d}w\\&=\frac{t^{\gamma-1+1/\alpha}}{\gamma}m_\Lambda^{(\alpha-1)/\alpha}\frac{\alpha}{\alpha-1}\,.\end{aligned}$$

Given (3.4.12), in order to prove (3.4.11) it suffices to show that

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty}\mathrm{P}\Big\{\,\exists n\in(\Lambda,\Lambda^{1+\epsilon})\;:\;F^{\leftarrow}\Big(1-\frac{1}{n}\Big)g_n\Big(\frac{N}{n}\Big)^{(1/\alpha)+\gamma-1}$$
$$|\Upsilon_\Lambda(n/N)-\mathrm{E}\Upsilon_\Lambda(n/N)|>Tg_{[0,n)}/k(\Lambda)\Big\}=0\,. \tag{3.4.13}$$

The scaling property we alluded to is the following.

Lemma 3.4.15. *For any positive real number λ, the processes $\Upsilon_\Lambda(\lambda\,\cdot)$ and $\lambda^{\gamma-1+(1/\alpha)}\Upsilon_\Lambda$ have the same distribution.*

Proof. We rewrite $\Upsilon_\Lambda(\lambda t)$ as

$$\begin{aligned}\int & (\lambda t - v)_+^{\gamma-1} w^{-1/\alpha} \mathbb{1}\{\, \lambda t w \leqslant m_\Lambda \,\} \,\mathrm{d}\Pi(v,w) \\ &= \lambda^{\gamma-1+1/\alpha} \int \Big(t - \frac{v}{\lambda}\Big)_+^{\gamma-1} (\lambda w)^{-1/\alpha} \mathbb{1}\{\, t\lambda w \leqslant m_\Lambda \,\} \,\mathrm{d}\Pi(v,w)\,.\end{aligned}$$

The image of the Poisson random measure Π by the map $(v,w) \mapsto (v/\lambda, \lambda w)$ is a Poisson random measure of intensity the Lebesgue measure, proving the lemma. ∎

In what follows we will use the following terminology. We say that a sequence of either functions or random variables, (f_n), converges to f in $\mathrm{L}^2(\mu)$-norm if $\lim_{n\to\infty} \int (f_n - f)^2 \,\mathrm{d}\mu$ converges to 0 as n tends to infinity. In our setting, (f_n) and f may not be in $\mathrm{L}^2(\mu)$ but $f_n - f$ is. If μ is the underlying probability P, we will write L^2 for $\mathrm{L}^2(\mathrm{P})$; in that case, convergence of the sequence of random variables (f_n) to f in L^2-norm means that $\lim_{n\to\infty} \mathrm{E}(f_n - f)^2 = 0$, again, even though f_n and f may not have finite variance but $f_n - f$ does.

Similarly, we will write that $f = g$ in L^2-norm, to mean $\mathrm{E}(f - g)^2 = 0$, even though f and g may not be square integrable.

Writing n/N as $(\Lambda/N)(n/\Lambda)$, Lemma 3.4.15 shows that, as a process indexed now by n in $(\Lambda, \Lambda^{1+\epsilon})$,

$$\Upsilon_\Lambda\Big(\frac{n}{N}\Big) \stackrel{\mathrm{d}}{=} \Big(\frac{\Lambda}{N}\Big)^{\gamma-1+(1/\alpha)} \Upsilon_\Lambda\Big(\frac{n}{\Lambda}\Big)\,.$$

Therefore, to prove (3.4.13) it suffices to show that

$$\begin{aligned}\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\, & \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, F^{\leftarrow}\Big(1 - \frac{1}{n}\Big) g_n \Big(\frac{\Lambda}{n}\Big)^{(1/\alpha)+\gamma-1} \\ & \Big| \Upsilon_\Lambda\Big(\frac{n}{\Lambda}\Big) - \mathrm{E}\Upsilon\Big(\frac{n}{\Lambda}\Big)\Big| > T g_{[0,n)}/k(\Lambda) \Big\} = 0\,,\end{aligned}$$

or, equivalently, that

$$\begin{aligned}\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\, & \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \,:\, \Big| \Upsilon_\Lambda\Big(\frac{n}{\Lambda}\Big) - \mathrm{E}\Upsilon_\Lambda\Big(\frac{n}{\Lambda}\Big)\Big| \\ & > T\Big(\frac{n}{\Lambda}\Big)^{(1/\alpha)+\gamma-1} \frac{k(n)}{k(\Lambda)} \Big\} = 0\,.\end{aligned}$$

Let η be a positive real number less than $(\alpha-1)/2\alpha$ and $1-(\alpha/2)$. Using Potter's bound, $k(n)/k(\Lambda) \gtrsim (n/\Lambda)^{1-(1/\alpha)-\eta}$ uniformly in n

between Λ and $\Lambda^{1+\epsilon}$ and as Λ tends to infinity. Thus to prove (3.4.13) it suffices to show that

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\Big\{ \exists n \in (\Lambda, \Lambda^{1+\epsilon}) \; : \\ \Big| \Upsilon_\Lambda\Big(\frac{n}{\Lambda}\Big) - \mathrm{E}\Upsilon_\Lambda\Big(\frac{n}{\Lambda}\Big)\Big| > T\Big(\frac{n}{\Lambda}\Big)^{\gamma-\eta} \Big\} = 0\,. \qquad (3.4.14)$$

Setting

$$f_{\Lambda,t}(v,w) = (t-v)_+^{\gamma-1} w^{-1/\alpha} \mathbb{1}\{ w \leqslant m_\Lambda/t \}\,,$$

we rewrite the centered version of Υ_Λ as a compensated Poisson integral,

$$(\Upsilon_\Lambda - \mathrm{E}\Upsilon_\Lambda)(t) = \int f_{\Lambda,t}(v,w)\,\mathrm{d}(\Pi - \mathrm{E}\Pi)(v,w)\,.$$

As Λ tends to infinity, the function $f_{\Lambda,t}$ converges pointwise to

$$f_t(v,w) = (t-v)_+^{\gamma-1} w^{-1/\alpha}\,.$$

This convergence holds in $\mathrm{L}^2(\,\mathrm{d}v\,\mathrm{d}w)$-norm since

$$\begin{aligned} \int (f_{\Lambda,t} - f_t)^2(v,w)\,\mathrm{d}v\,\mathrm{d}w \\ &= \int (t-v)_+^{2(\gamma-1)} w^{-2/\alpha} \mathbb{1}\{ w \geqslant m_\Lambda/t \}\,\mathrm{d}v\,\mathrm{d}w \\ &= t^{2(\gamma-1+1/\alpha)} m_\Lambda^{-(2/\alpha)+1} \frac{\alpha}{(2-\alpha)(2\gamma-1)}\,. \qquad (3.4.15) \end{aligned}$$

Recall that the compensated Poisson integral induces an isometry in the sense that for any function f in $\mathrm{L}^2(\,\mathrm{d}v\,\mathrm{d}w)$

$$\mathrm{E}\Big(\int f\,\mathrm{d}(\Pi - \mathrm{E}\Pi)\Big)^2 = \int f^2(v,w)\,\mathrm{d}v\,\mathrm{d}w\,.$$

It then follows from (3.4.15) that $\lim_{\Lambda\to\infty} \Upsilon_\Lambda - \mathrm{E}\Upsilon_\Lambda$ exists pointwise in L^2-norm and is the compensated Poisson integral

$$\Upsilon_0(t) = \int f_t\,\mathrm{d}(\Pi - \mathrm{E}\Pi)\,.$$

Our next lemma implies that we can replace $\Upsilon_\Lambda - \mathrm{E}\Upsilon_\Lambda$ by its limit Υ_0 in 3.4.1.

Lemma 3.4.16. *Provided β is positive, we have*

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty} \mathrm{P}\Big\{\,\exists n\in(\Lambda,\Lambda^{1+\epsilon})\,:\,\Big|\int (f_{\Lambda,n/\Lambda}-f_{n/\Lambda})\,\mathrm{d}(\Pi-\mathrm{E}\Pi)\Big| \geqslant T\Big(\frac{n}{\Lambda}\Big)^{\gamma-\eta}\Big\}=0\,.$$

Proof. Since

$$(f_t-f_{\Lambda,t})(v,w)=(t-v)_+^{\gamma-1}w^{-1/\alpha}\mathbb{1}\{\,w\geqslant m_\Lambda/t\,\}$$

is in $\mathrm{L}^p(\,\mathrm{d}v\,\mathrm{d}w)$ for any p greater than α, let

$$M_p=\mathrm{E}\Big(\int (f_{\Lambda,t}-f_t)\,\mathrm{d}(\Pi-\mathrm{E}\Pi)\Big)^p\,.$$

In what follows we restrict p to be an integer. Using Privault's (2009, equation 9; or 2010, equation 2.9) moment identity (see also Bassan and Bona, 1990), we have for p at least 2,

$$M_p=\sum_{0\leqslant k\leqslant p-2}\binom{p-1}{k}\int (f_{\Lambda,t}-f_t)^{p-k}(v,w)\,\mathrm{d}v\,\mathrm{d}w M_k\,. \tag{3.4.16}$$

We now prove by induction that for some constant c_p,

$$|M_p|\leqslant c_p t^{p(\gamma-1+1/\alpha)}m_\Lambda^{-(p/\alpha)+\lfloor p/2\rfloor}\,. \tag{3.4.17}$$

Indeed, M_1 vanishes and, as shown in (3.4.15),

$$M_2=c_2t^{2(\gamma-1+1/\alpha)}m_\Lambda^{-(2/\alpha)+1}\,.$$

Assume that for any k less than p,

$$|M_k|\leqslant c_k t^{k(\gamma-1+1/\alpha)}m_\Lambda^{-(k/\alpha)+\lfloor k/2\rfloor}\,.$$

Since

$$\int (f_t-f_{\Lambda,t})^{p-k}(v,w)\,\mathrm{d}v\,\mathrm{d}w=ct^{(p-k)(\gamma-1+1/\alpha)}m_\Lambda^{1-((p-k)/\alpha)}\,,$$

equality (3.4.16) and the induction hypothesis imply

$$|M_p|\leqslant c\sum_{0\leqslant k\leqslant p-2}t^{(p-k)(\gamma-1+1/\alpha)}m_\Lambda^{1-(p-k)/\alpha} \times t^{k(\gamma-1+1/\alpha)}m_\Lambda^{-(k/\alpha)+\lfloor k/2\rfloor}\,.$$

In this sum, bounding $m_\Lambda^{-(k/\alpha)+\lfloor k/2\rfloor}$ by $m_\Lambda^{-(k/\alpha)+\lfloor (p-2)/2\rfloor}$ yields

$$|M_p| \leqslant c_p t^{p(\gamma-1+1/\alpha)} m_\Lambda^{1-(p/\alpha)+\lfloor (p-2)/2\rfloor}\,,$$

which is (3.4.17).

We then take p to be an even integer. Applying Markov's inequality for any n between Λ and $\Lambda^{1+\epsilon}$ and using (3.4.17),

$$\begin{aligned} \mathrm{P}\Big\{ \Big|\int (f_{\Lambda,n/\Lambda} - f_{n/\Lambda})\,\mathrm{d}(\Pi-\mathrm{E}\Pi)\Big| \geqslant T\Big(\frac{n}{\Lambda}\Big)^{\gamma-\eta} \Big\} & \\ \leqslant T^{-p}\Big(\frac{\Lambda}{n}\Big)^{(\gamma-\eta)p} c_p \Big(\frac{n}{\Lambda}\Big)^{p(\gamma-1+1/\alpha)} m_\Lambda^{-(p/\alpha)+\lfloor p/2\rfloor} & \\ \leqslant c_p T^{-p}\Big(\frac{\Lambda}{n}\Big)^{p(1-(1/\alpha)-\eta)} m_\Lambda^{-p((1/\alpha)-(1/2))}\,. & \end{aligned}$$

Applying Bonferroni's inequality, the probability involved in the lemma is at most

$$c_p T^{-p}\Lambda^{p(1-1/\alpha-\eta)} m_\Lambda^{-p((1/\alpha)-(1/2))} \sum_{\Lambda\leqslant n\leqslant \Lambda^{1+\epsilon}} n^{-p(1-(1/\alpha)-\eta)}\,.$$

Taking p larger than $1/\big(1-(1/\alpha)-\eta\big)$, this bound is of order

$$\begin{aligned} cT^{-p}\Lambda^{p(1-(1/\alpha))} m_\Lambda^{-p(1-(1/\alpha)-(1/2))}\Lambda^{1-p(1-(1/\alpha))} & \\ = cT^{-p}\Lambda m_\Lambda^{-p((1/\alpha)-(1/2))}\,. & \end{aligned}$$

This bound is regularly varying in Λ of index

$$1-\beta p\Big(\frac{1}{\alpha}-\frac{1}{2}\Big)\,.$$

Thus, taking p larger than $2\alpha/\big(\beta(2-\alpha)\big)$, it tends to 0 as Λ tends to infinity, proving the lemma. ∎

Given Lemma 3.4.16, we see that to prove (3.4.14) it suffices to show that

$$\lim_{T\to\infty}\limsup_{\Lambda\to\infty} \mathrm{P}\Big\{ \exists n\in(\Lambda,\Lambda^{1+\epsilon}) \,:\, |\Upsilon_0(n/\Lambda)| > T(n/\Lambda)^{\gamma-\eta} \Big\} = 0\,. \tag{3.4.18}$$

Our next step is to identify the process Υ_0 as a fractional integral of a spectrally positive Lévy stable process.

Consider the spectrally positive centered Lévy stable process given by its Itô representation

$$L_0^+(t) = \int \mathbb{1}_{[0,t]}(v) w^{-1/\alpha} \, \mathrm{d}(\Pi - \mathrm{E}\Pi)(v,w) \, .$$

This process is the pointwise limit in L^2-norm of

$$L_{0,\epsilon}^+(t) = \int \mathbb{1}_{[0,t]}(v) w^{-1/\alpha} \mathbb{1}_{[0,1/\epsilon]}(w) \, \mathrm{d}(\Pi - \mathrm{E}\Pi)(v,w)$$

as ϵ tends to 0. Moreover, defining

$$\begin{aligned} L_{0,\epsilon}^{+(\gamma-1)}(t) &= \int \gamma(t-v)_+^{\gamma-1} \, \mathrm{d}L_{0,\epsilon}^+(v) \\ &= \sum_{i\geqslant 1} \Big(\gamma(t-V_i)_+^{\gamma-1} W_i^{-1/\alpha} \mathbb{1}_{[0,1/\epsilon]}(W_i) \\ &\qquad\qquad\qquad - t^\gamma \mathrm{E} W_i^{-1/\alpha} \mathbb{1}_{[0,1/\epsilon]}(W_i) \Big) \\ &= \int \gamma(t-v)_+^{\gamma-1} w^{-1/\alpha} \mathbb{1}_{[0,1/\epsilon]}(w) \, \mathrm{d}(\Pi - \mathrm{E}\Pi)(v,w) \, , \end{aligned}$$

we see that $L_{0,\epsilon}^{+(\gamma-1)}$ converges pointwise in L^2-norm to $\gamma\Upsilon_0$ as ϵ tends to 0. It follows that pointwise in L^2-norm,

$$\Upsilon_0^+(t) = \int (t-v)_+^{\gamma-1} \, \mathrm{d}L_0^+(v) \, . \tag{3.4.19}$$

Lemma 3.1.7 in Barbe and McCormick (2010) implies that the right hand side of (3.4.19) is almost surely continuous. Considering the fractional integral

$$L_0^{+(\gamma-1)}(t) = \int \gamma(t-v)_+^{\gamma-1} \, \mathrm{d}L_0^+(v) \, ,$$

and since $\Upsilon_0^+(n/\Lambda)$ and $L_0^{+(\gamma-1)}(n/\Lambda)/\gamma$ coincide in L^2-norm for any integer n between Λ and $\Lambda^{1+\epsilon}$, we see that in order to prove (3.4.18) and therefore Proposition 3.4.1, it suffices to show the following.

Lemma 3.4.17. *For any positive η sufficiently small,*

$$\lim_{T\to\infty} \mathrm{P}\{\, \exists t \geqslant 1 \, : \, |L_0^{+(\gamma-1)}(t)| \geqslant T t^{\gamma-\eta} \,\} = 0 \, .$$

Proof. Note that L_0^+ vanishes at 0. For any nonnegative t, the function $(t-\mathrm{Id})_+^{\gamma-1}$ is deterministic, differentiable on $[0,t]$, so that its quadratic covariation with L_0^+ vanishes on $[0,t]$. We then integrate by parts the integral defining $L_0^{+(\gamma-1)}$ (see Protter, 1992, chapter 2.6, Corollary 2) as

$$L_0^{+(\gamma-1)}(t) = \gamma(\gamma-1)\int L_0^+(v)(t-v)_+^{\gamma-2}\,\mathrm{d}v.$$

It follows from Pruitt (1981) that there exists a constant c and a random v_0 such that $|L_0^+(v)| \leqslant cv^{(1/\alpha)+\eta}$ for any v at least v_0. This implies that for t at least v_0,

$$\int_{v_0}^t |L_0^+(v)|(t-v)_+^{\gamma-2}\,\mathrm{d}v \leqslant ct^{\gamma-1+(1/\alpha)+\eta}\,.$$

Since

$$\int_0^{v_0} |L_0^+(v)|(t-v)_+^{\gamma-2}\,\mathrm{d}v \leqslant c\sup_{0\leqslant v\leqslant v_0}|L_0^+(v)|t^{\gamma-1}$$

and $-1+(1/\alpha)+2\eta$ is negative for η small enough, we have

$$\lim_{t\to\infty} t^{-\gamma+\eta}L_0^{+(\gamma-1)}(t) = 0$$

almost surely. The lemma follows. ■

3.5. Concluding the proof. Having completed the proof of Proposition 3.4.1, we can complete that of Theorem 2.2.

First, we settle the assertion that

$$\sup_{t\geqslant 0} L_0^{(\gamma-1)}(t) - t^\gamma \text{ is almost surely finite.} \tag{3.5.1}$$

Let $\widetilde{L}_0^{+(\gamma-1)}$ be an independent copy of $L_0^{+(\gamma-1)}$. It is shown in section 3.3 of Barbe and McCormick (2010) that $L_0^{(\gamma-1)}$ has the same distribution as $p^{1/\alpha}L_0^{+(\gamma-1)} - q^{1/\alpha}\widetilde{L}_0^{+(\gamma-1)}$. Consequently, (3.5.1) follows from Lemma 3.4.17.

To complete the proof of Theorem 2.2, we set

$$T_n^- = \sum_{0\leqslant i<n} g_iX_{n-i}\mathbb{1}\{\,X_{n-i}\leqslant a_{n,\Lambda}\,\}\,.$$

Note that

$$T_n^- = -\sum_{0\leqslant i<n} g_i(-X_{n-i})\mathbb{1}\{\, X_{n-i} \geqslant -a_{n,\Lambda} \,\}\,.$$

If assumption (2.6) hold, we substitute $\mathsf{M}_{-1}F$ for F in steps two, three and four to obtain that

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\{\, \exists n \geqslant \Lambda \,:\, |T_n^- - \mathrm{E}T_n^-| > Tg_{[0,n)}/k(\Lambda) \,\} = 0\,.$$

This, combined with Propositions 3.1.1, 3.2.1, 3.3.1 and 3.4.1 give our heavy traffic approximation under (2.6).

In order to prove Theorem 2.2 under (2.7) we will use a coupling argument based on a decomposition of the innovations into a part which is bounded from above and a part which is bounded from below. The underlying idea is that the part bounded from above should not contribute too much to the process S_n reaching the boundary $ag_{[0,n)}$. To make this argument viable, note that combining Propositions 3.1.1, 3.2.1, 3.3.1 and 3.4.1, Theorem 2.2 holds when F is supported on some interval bounded away from $-\infty$.

The following lemma allows us to define the proper representation of the innovations. If G is a distribution function, $G^{\leftarrow}(0+)$ and $G^{\leftarrow}(1)$ are respectively the lower and upper end point of the support of the underlying probability measure.

Lemma 3.5.1. *Let F be a distribution function on the real line, centered and such that $\overline{F}$ is regularly varying. There exists two distribution functions F_L and F_U and a r in $(0,1)$ such that*

(i) $F_U^{\leftarrow}(0+) > -\infty$ and $F_L^{\leftarrow}(1) < \infty$,

(ii) F_U and F_L are centered,

(iii) $F = rF_U + (1-r)F_L$.

Given assertions (i) and (iii), we must have $\overline{F} = r\overline{F}_U$ and $\overline{\mathsf{M}_{-1}F} = (1-r)\overline{\mathsf{M}_{-1}F_L}$ ultimately. Thus rF_U and $(1-r)F_L$ capture respectively the upper and lower tails of F.

Proof. Since F has a mean, define $A(t) = \int_{(t,\infty)} x \,\mathrm{d}F(x)$ and set $\theta = A(0)/2$. The function A is nonincreasing and càdlàg on the nonnegative half-line. Let

$$t_1 = \inf\{\, t \,:\, A(t) < \theta \,\}\,.$$

If $A(t_1) < \theta$, then F has a jump at t_1 and there exists a positive τ_1 at most $F(t_1) - F(t_1-)$ such that $A(t_1) + \tau_1 t_1 = \theta$; otherwise we set $\tau_1 = 0$.

Similarly, define $B(t) = \int_{(t,0]} x \,\mathrm{d}F(x)$. The function B is nondecreasing on the negative half-line. Since F is centered, $\lim_{t\to-\infty} B(t) = -A(0)$. Let

$$t_2 = \sup\{\, t : B(t) < -\theta \,\} .$$

If $B(t_2) > -\theta$ then F has a jump at t_2 and we define τ_2 such that $B(t_2-) + \tau_2 t_2 = -\theta$; otherwise, we set $\tau_2 = 0$.

We define the measure μ_U by

$$\frac{\mathrm{d}\mu_U}{\mathrm{d}F} = \mathbb{1}_{(t_2,0]} + \mathbb{1}_{(t_1,\infty)} + \tau_1 \mathbb{1}_{\{t_1\}} + \tau_2 \mathbb{1}_{\{t_2\}} ,$$

and set $F_U = \mu_U / \mu_U(\mathbb{R})$. By construction F_U is centered. Set $r = \mu_U(\mathbb{R})$ and define $F_L = (F - rF_U)/(1-r)$. ∎

Our next lemma, valid since α is positive and less than 2, relates the moment generating function of F_L to the bound (2.7). It is convenient to define

$$D(t) = \frac{c}{1-r} \overline{F}(t\log t) \log t \,,$$

r being as in Lemma 3.5.1 and the constant c being, for once, the same as in (2.7)

Lemma 3.5.2. *The following holds as λ tends to 0 from above,*

$$\int e^{\lambda x} \,\mathrm{d}F_L(x) \leqslant 1 + D\Big(\frac{1}{\lambda}\Big) \int_0^\infty \frac{1-e^{-u}}{u^\alpha} \,\mathrm{d}u \big(1 + o(1)\big) \,.$$

Proof. Since F_L is centered, two integrations by parts yield

$$0 = \int x \,\mathrm{d}F_L(x) = -\int_{-\infty}^0 F_L(x) \,\mathrm{d}x + \int_0^\infty \overline{F}_L(x) \,\mathrm{d}x \,.$$

This identity, an integration by parts and considering that $\overline{F}_L$ vanishes ultimately yield that the moment generating function of

F_L is

$$\lambda \int e^{\lambda x} \overline{F}_L(x)\,\mathrm{d}x$$
$$= 1 - \lambda \int_{-\infty}^{0} e^{\lambda x} F_L(x)\,\mathrm{d}x + \lambda \int_0^\infty e^{\lambda x} \overline{F}_L(x)\,\mathrm{d}x$$
$$= 1 + \lambda \int_{-\infty}^{0} (1 - e^{\lambda x}) F_L(x)\,\mathrm{d}x + \lambda \int_0^\infty (e^{\lambda x} - 1) \overline{F}_L(x)\,\mathrm{d}x\,.$$

Let t_0 be a negative real number such that $(1-r)F_L$ coincides with F on $(-\infty, t_0\,]$ and and $\overline{M_{-1}F} \leqslant (1-r)D$ on $[\,t_0, \infty)$. Since $\overline{F}_L$ vanishes ultimately,

$$\lambda \int_{t_0}^{0} (1 - e^{\lambda x}) F_L(x)\,\mathrm{d}x + \lambda \int_0^\infty (e^{\lambda x} - 1) \overline{F}_L(x)\,\mathrm{d}x = O(\lambda^2)$$

as λ tends to 0. Hence,

$$\int e^{\lambda x}\,\mathrm{d}F_L(x) \leqslant 1 + \lambda \int_{-\infty}^{t_0} (1 - e^{\lambda x}) D(-x)\,\mathrm{d}x + O(\lambda^2)$$
$$= 1 + \int_{-\infty}^{\lambda t_0} (1 - e^{y}) D(-y/\lambda)\,\mathrm{d}y + O(\lambda^2)\,. \quad (3.5.2)$$

Since D is regularly varying of index $-\alpha$ less than -1, the integral in (3.5.2) is asymptotically equivalent to

$$D\Big(\frac{1}{\lambda}\Big) \int_{-\infty}^{0} \frac{1 - e^y}{(-y)^\alpha}\,\mathrm{d}y\,.$$

Since α is between 1 and 2, $D(1/\lambda) \gg \lambda^2$ as λ tends to 0 and the lemma follows. ∎

To conclude the proof of Theorem 2.2, write $F = rF_U + (1-r)F_L$ for the decomposition given in Lemma 3.5.1. Let $(X_{U,i})$ and $(X_{L,i})$ be two independent sequences of independent random variables distributed respectively according to F_U and F_L. Let (B_i) be an independent sequence of independent random variables having a Bernoulli distribution with parameter p. Set

$$X_i = B_i X_{U,i} + (1 - B_i) X_{L,i}\,.$$

By construction (X_i) is a sequence of independent random variables all having distribution F. Let (S_n), $(S_{U,n})$ and $(S_{L,n})$ by the (g,F)-, (g,F_U)- and (g,F_L)-processes based on (X_i), $(X_{U,i})$ and $(X_{L,i})$ respectively. We have $S_n = S_{L,n} + S_{U,n}$. Hence, refering to (3.6),

$$\begin{aligned} \mathrm{P}\{\, \exists n \geqslant \Lambda \,:\, S_n > 2Tg_{[0,n)}/k(\Lambda)\,\} \hspace{-4cm} & \\ & \leqslant \mathrm{P}\{\, \exists n \geqslant \Lambda \,:\, S_{L,n} > Tg_{[0,n)}/k(\Lambda)\,\} \\ & \qquad + \mathrm{P}\{\, \exists n \geqslant \Lambda \,:\, S_{U,n} > Tg_{[0,n)}/k(\Lambda)\,\} \end{aligned}$$

Using the part of Theorem 2.2 that we proved already, more precisely when the innovations are bounded from below,

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \mathrm{P}\{\, \exists n \geqslant \Lambda \,:\, S_{U,n} > Tg_{[0,n)}/k(\Lambda)\,\} = 0\,.$$

Hence, using a Bonferroni inequality, it suffices to show that

$$\lim_{T\to\infty} \limsup_{\Lambda\to\infty} \sum_{n\geqslant\Lambda} \mathrm{P}\{\, S_{L,n} > Tg_{[0,n)}/k(\Lambda)\,\} = 0\,. \tag{3.5.3}$$

For this, let φ be the moment generating function of $(1-B_i)X_{L,i}$ and let φ_L be that of $X_{L,i}$. We have

$$\varphi(t) = \mathrm{E}e^{t(1-B_i)X_{L,i}} = r + (1-r)\varphi_L(t)\,. \tag{3.5.4}$$

The exponential form of Markov's inequality yields for any positive λ

$$\mathrm{P}\{\, S_{L,n} > Tg_{[0,n)}/k(\Lambda)\,\} \leqslant \exp\Bigl(-\lambda T\frac{g_{[0,n)}}{k(\Lambda)} + \sum_{0\leqslant i<n} \log\varphi(\lambda g_i)\Bigr)\,. \tag{3.5.5}$$

In this inequality we take $\lambda = k(n)(\log n)/g_{[0,n)}$ and proceed to estimate (3.5.5).

Since k is regularly varying of positive index, $k(\Lambda) \leqslant ck(n)$ if $n \geqslant \Lambda$ and Λ is large enough. Consequently, $\lambda Tg_{[0,n)}/k(\Lambda) > cT$.

Since (3.1) holds, our choice of λ implies

$$\lambda \sim c\frac{\log n}{F^{\leftarrow}(1-1/n)g_n}\,;$$

thus, since (g_i) is asymptotically equivalent to a nondecreasing sequence, the bound

$$\max_{0\leqslant i<n} \lambda g_i \leqslant c\frac{\log n}{F^{\leftarrow}(1-1/n)}$$

shows that this maximum tends to 0 as Λ tends to infinity and n is at least Λ. Then, considering (3.5.4) and Lemma 3.5.2 to bound φ_L, we obtain that for Λ large enough, n at least Λ and any i between 0 and n,

$$\begin{aligned}\log\varphi(\lambda g_i) &\leqslant \log\Big(r+(1-r)\big(1+cD(1/\lambda g_i)\big)\Big)\\ &= \log\big(1+cD(1/\lambda g_i)\big)\\ &\leqslant cD(1/\lambda g_i)\,.\end{aligned}$$

Since D is regularly varying of index $-\alpha$, this implies that (3.5.5) is at most

$$\exp\Big(-Tc\log n + cnD\Big(\frac{F^{\leftarrow}(1-1/n)}{\log n}\Big)\Big)\,. \tag{3.5.6}$$

We claim that

$$D\Big(\frac{F^{\leftarrow}(1-1/n)}{\log n}\Big) \leqslant c\frac{\log n}{n}\,. \tag{3.5.7}$$

Indeed, setting $s = F^{\leftarrow}(1-1/n)/\log n$, we have $s\log s \sim cF^{\leftarrow}(1-1/n)$, which implies $\overline{F}(s\log s) \sim c/n$. Moreover $\log s \sim c\log n$ as n tends to infinity. This allows us to write (3.5.7) as $D(s) \leqslant c\log s\overline{F}(s\log s)$ which is true by definition of D.

Combining (3.5.6) and (3.5.7), we see that, if T is large enough, (3.5.5) is at most n^{-2}. This proves (3.5.3) and completes our proof of Theorem 2.2.

3.6. On assumption (2.5). To explain the role of assumption (2.5), we first explain that of (m_n). The proof of Propositions 3.1.1, 3.3.1 and Lemma 3.4.16 are similar: we evaluate some moment of high order and use Bonferroni's inequality. In order for the Bonferroni bound to tend to 0, we need (m_n) grow to infinity at least at an algebraic rate.

In step 2 we do not rely on (2.5) but on (3.2.2). If (m_n) were allowed to be slowly varying, step 2 would sill hold. The same is true for the approximations of the $T^+_{i,n,N}$ in step 4, with the caveat that for Lemma 3.4.3 to hold, we need to have $\liminf_{n\to\infty} m_n/\log n > 0$, but this is essentially implied by (3.2.3).

That (m_n) has to grow at an algebraic rate makes (3.2.2) equivalent to (2.5). So we see that the only reason (2.5) is needed is because our crude estimate in the proofs of Propositions 3.1.1, 3.3.1 and Lemma 3.4.16.

If we do not take those propositions and this lemma into account, the condition (3.2.3) would allow $m_n = (\log n)^{1+\epsilon}$. To obtain an even slower rate requires not to rely on Kiefer's theorem. It is conceivable that a good description of the extremes $(U_i \mathbb{1}\{\, U_i > 1 - m_n/n \,\})_{1\leqslant i\leqslant n}$ as n changes could yield a better result with no other condition than (2.5). This seems related to the asymptotic behavior of m_n-records, but our attempt to devise a proof in this direction failed.

Note that assumption (2.3) was used only in the proof of Lemma 3.4.10 and 3.4.13. A close examination of the proofs shows that we could replace (2.3) by the existence of a sequence (ρ_n) such that

$$\limsup_{n\to\infty} \sup_{1/\rho_n \leqslant i/n \leqslant 1} \Bigl| \frac{g_i}{g_n} - \Bigl(\frac{i}{n}\Bigr)^{\gamma-1} \Bigr| < \infty$$

and $\rho_n \geqslant \xi_n^{1/\alpha}(m_n \vee \log n)$. So, any improvement on the rate of growth of (m_n) would translate into an improvement of (2.3) as well.

4. Proof of Proposition 2.1. *(i)⇒(ii).* Write $F^{\leftarrow}(1-1/t) = t^{1/\alpha}\ell(t)$ where ℓ is regularly varying. Note that ℓ is ultimately positive. If (i) holds then, using $1/\lambda$ instead of λ,

$$\lim_{t\to\infty} t^{\kappa} \sup_{t^{-\kappa}\leqslant\lambda\leqslant t^{\kappa}} \lambda^{1/\alpha} \Bigl| \frac{\ell(\lambda t)}{\ell(t)} - 1 \Bigr| = 0\,.$$

In particular, for any fixed λ greater than 1, we have, for any t large enough,

$$\ell(t)(1-t^{-\kappa}) \leqslant \ell(\lambda t) \leqslant \ell(t)(1+t^{-\kappa})\,.$$

Hence, for any positive integer n,

$$\ell(\lambda^{n-1}t)\bigl(1-(\lambda^{n-1}t)^{-\kappa}\bigr) \leqslant \ell(\lambda^n t) \leqslant \ell(\lambda^{n-1}t)\bigl(1+(\lambda^{n-1}t)^{-\kappa}\bigr)\,.$$

By induction, this implies

$$\ell(t) \prod_{0\leqslant i<n} \bigl(1-(\lambda^i t)^{-\kappa}\bigr) \leqslant \ell(\lambda^n t) \leqslant \ell(t) \prod_{0\leqslant i<n} \bigl(1+(\lambda^i t)^{-\kappa}\bigr)\,.$$

Since ℓ is slowly varying,

$$\lim_{n\to\infty} \sup_{s\in[\lambda^n t, \lambda^{n+1}t]} \frac{\ell(s)}{\ell(\lambda^n t)} = 1\,.$$

Therefore,

$$\ell(t)\prod_{i\geqslant 0}\bigl(1-(\lambda^i t)^{-\kappa}\bigr)\leqslant \liminf_{s\to\infty}\ell(s)$$
$$\leqslant \limsup_{s\to\infty}\ell(s)\leqslant \ell(t)\prod_{i\geqslant 0}\bigl(1+(\lambda^i t)^{-\kappa}\bigr)\,. \quad (4.1)$$

Since

$$\lim_{t\to\infty}\prod_{i\geqslant 0}\bigl(1-(\lambda^i t)^{-\kappa}\bigr)=\lim_{t\to\infty}\prod_{i\geqslant 0}\bigl(1+(\lambda^i t)^{-\kappa}\bigr)=1\,,$$

we obtain

$$\limsup_{t\to\infty}\ell(t)\leqslant \liminf_{s\to\infty}\ell(s)\leqslant \limsup_{s\to\infty}\ell(s)\leqslant \liminf_{t\to\infty}\ell(t)\,,$$

proving that $\lim_{t\to\infty}\ell(t)$ exists. This limit is then positive due to (4.1). Therefore, there exists a function $\delta(\cdot)$ which tends to 0 at infinity such that

$$F^{\leftarrow}(1-1/t)=ct^{1/\alpha}\bigl(1+\delta(t)\bigr)\,.$$

We can then rewrite (i) as

$$\lim_{t\to\infty}t^{\kappa}\sup_{t^{-\kappa}\leqslant\lambda\leqslant t^{\kappa}}\lambda^{1/\alpha}\Bigl|\frac{1+\delta(\lambda t)}{1+\delta(t)}-1\Bigr|=0\,. \quad (4.2)$$

Since $\delta(\cdot)$ tends to 0 at infinity, (4.2) implies

$$\lim_{t\to\infty}t^{\kappa}\sup_{t^{-\kappa}\leqslant\lambda\leqslant t^{\kappa}}\lambda^{1/\alpha}|\delta(\lambda t)-\delta(t)|=0\,.$$

In particular, if ϵ is a fixed positive real number, for any t large enough, considering $\lambda=t^{\kappa}$,

$$|\delta(t^{1+\kappa})-\delta(t)|\leqslant \epsilon t^{-\kappa(1+1/\alpha)}\,.$$

Substituting $t^{(1+\kappa)^n}$ for t, we obtain

$$|\delta(t^{(1+\kappa)^{(n+1)}})-\delta(t^{(1+\kappa)^n})|\leqslant \epsilon t^{-\kappa(1+\kappa)^n(1+1/\alpha)}\,.$$

Since $\delta(\cdot)$ tends to 0 at infinity, summing all these inequalities over n nonnegative, we obtain

$$|\delta(t)|\leqslant \epsilon t^{-\kappa(1+1/\alpha)}+ct^{-\kappa(1+\kappa)(1+1/\alpha)}\,.$$

Consequently, $\delta(t)=o(t^{-\kappa(1+1/\alpha)})$ and (ii) holds.

(ii)⇒(i). Whenever $t\wedge(\lambda t)$ tends to infinity,

$$\begin{aligned}\frac{F^{\leftarrow}(1-1/\lambda t)}{F^{\leftarrow}(1-1/t)}-\lambda^{1/\alpha}&=\lambda^{1/\alpha}\Bigl(\frac{1+o\bigl((\lambda t)^{-\kappa(1+1/\alpha)}\bigr)}{1+o(t^{-\kappa(1+1/\alpha)})}-1\Bigr)\\&=\lambda^{1/\alpha}\Bigl(o\bigl((\lambda t)^{-\kappa(1+1/\alpha)}\bigr)\vee o(t^{-\kappa(1+1/\alpha)})\Bigr).\end{aligned}$$

In particular, since κ is less than both 1 and $2/(\alpha+1)$,

$$\sup_{t^{-\kappa}\leqslant\lambda\leqslant t^{\kappa}}\Bigl|\frac{F^{\leftarrow}(1-1/\lambda t)}{F^{\leftarrow}(1-1/t)}-\lambda^{1/\alpha}\Bigr|=o(t^{-\kappa})$$

and (i) holds.

(ii)⇒(iii). Consider the relation $x=F^{\leftarrow}(1-1/t)$. Given (ii), this means, as either t or x tend to infinity,

$$x=ct^{1/\alpha}\bigl(1+o(t^{-\kappa(1+1/\alpha)})\bigr). \tag{4.3}$$

The proof is then an easy exercise in asymptotic analysis. In particular, $x\sim ct^{1/\alpha}$ and $t\sim(x/c)^{\alpha}$. Write $t=(x/c)^{\alpha}(1+y)$ where y tends to 0 as t tends to infinity. Using (4.3) we obtain

$$\begin{aligned}x&=x(1+y)^{1/\alpha}\bigl(1+o(x^{-\kappa(\alpha+1)})\bigr)\\&=x+\frac{1}{\alpha}xy+o(x^{1-\kappa(\alpha+1)})+O(xy^2).\end{aligned}$$

Therefore, $y=o(x^{-\kappa(\alpha+1)})+O(y^2)$. This forces $y=o(x^{-\kappa(\alpha+1)})$. Thus,

$$t=(x/c)^{\alpha}\bigl(1+o(x^{-\kappa(\alpha+1)})\bigr). \tag{4.4}$$

Since $x=F^{\leftarrow}(1-1/t)$ and F is continuous, we have $\overline{F}(x)=1/t$ as t tends to infinity. Given (4.4), this implies

$$\overline{F}(x)=(c/x)^{\alpha}\bigl(1+o(x^{-\kappa(\alpha+1)})\bigr),$$

that is (iii).

(iii)⇒(ii). The proof is similar to that of its converse.

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